MATH 172 MIDTERM EXAM (05.11.2009)

Q1			Q2			Q3	Q4		Total

Name: **No:**
Instructor:
Section: **Make sure that you write your section number!**

ATTENTION: There are 4 questions on 4 pages. Solve all of them. Duration is 60 minutes. Simply giving a final result is not sufficient to answer any question, so show all the steps you pursued to get any final result. Otherwise your answer will not be evaluated as a correct answer.

1. Evaluate the following integrals. (10 points each)

(a) $\int (4-3z)^4\,dz$

Solution:

$$\int (4-3z)^4\,dz = \int u^4 \frac{du}{-3} = -\frac{1}{3}\frac{u^5}{5} + C = -\frac{1}{15}(4-3z)^5 + C \qquad \begin{cases} u = 4-3z \\ du = -3dz \end{cases}$$

(b) $\int 6x^2 e^{x^3-2}\,dx$

Solution:

$$\int 6x^2 e^{x^3-2}\,dx = \int 2e^u\,du = 2e^u + C = 2e^{x^3-2} + C \qquad \begin{cases} u = x^3-2 \\ du = 3x^2dx \end{cases}$$

(c) $\int \left(\frac{1-2x^5}{x^3}\right) dx$

Solution:

$$\int \left(\frac{1-2x^5}{x^3}\right) dx = \int \left(\frac{1}{x^3} - 2x^2\right) dx = \int x^{-3}dx - 2\int x^2 dx = -\frac{1}{2x^2} - \frac{2}{3}x^3 + C$$

2. Find the following integrals (10 points each)

(a) $\int_0^1 \frac{x+1}{e^x}\,dx$

Solution:

$$\int_0^1 \frac{x+1}{e^x}dx = -(x+1)e^{-x}\Big|_0^1 - \int_0^1 -e^{-x}dx \qquad \begin{cases} u = x+1 & \Rightarrow \; du = dx \\ dv = e^{-x}dx & \Rightarrow \; v = -e^{-x} \end{cases}$$

$$= -2e^{-1} + 1 - e^{-x}\Big|_0^1 = -2e^{-1} + 1 - e^{-1} + 1 = 2 - 3e^{-1}$$

(b) $\int \frac{\ln(x+1)}{x+1}dx$

Solution:

$$\int (x+1)\ln(x+1)dx = \frac{1}{2}(x+1)^2\ln(x+1) - \int \frac{1}{2}(x+1)^{\not{2}} \frac{1}{\not{x+1}}dx \begin{cases} u = \ln(x+1) & \Rightarrow \; du = \frac{1}{x+1}dx \\ dv = (x+1)dx & \Rightarrow \; v = \frac{1}{2}(x+1)^2 \end{cases}$$

$$= \frac{1}{2}(x+1)^2\ln(x+1) - \frac{1}{4}(x+1)^2 + C$$

$$= \frac{1}{4}(x+1)^2\left(2\ln(x+1) - 1\right) + C$$

(c) $\int \frac{5x^2+2}{x^3+x}dx$

Solution:

First we have to express the rational function in terms of partial fraction:

$$\frac{5x^2+2}{x^3+x} = \frac{5x^2+2}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x^3+x}$$

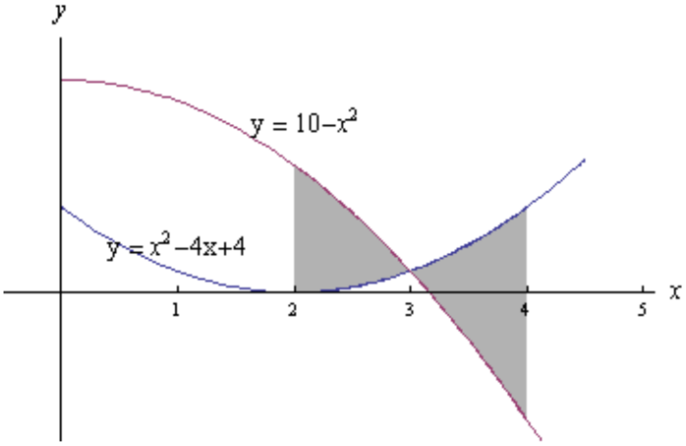
$$5x^2+2 = \left(A+B\right)x^2 + Cx + A$$

$$C = 0; \quad A = 2; \quad A+B = 5 \qquad \Rightarrow \qquad B = 3$$

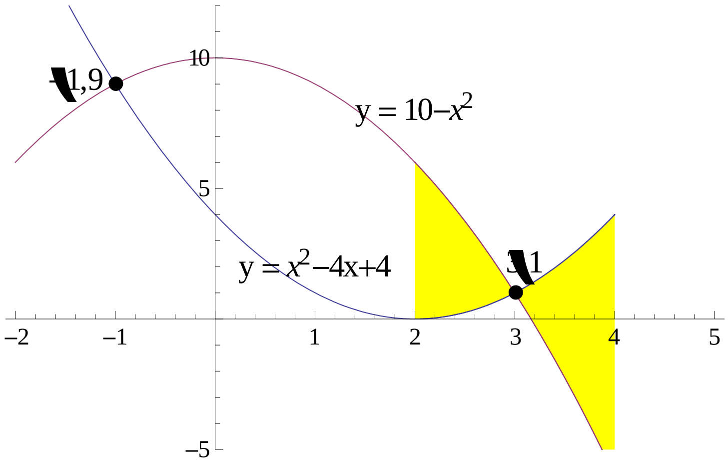
$$\int \frac{5x^2+2}{x^3+x}dx = \int \frac{2}{x}dx + \int_1^e \frac{3x}{\underbrace{x^2+1}_{u}}dx = 2\ln|x| + 3\int u^{-1}\frac{du}{2}$$

$$= 2\ln|x| + \frac{3}{2}\ln(x^2+1) + C = \ln\left[x^2(x^2+1)^{3/2}\right] + C$$

3. Express the area of the shaded region in terms of an integral (or integrals) and evaluate. (You have to find the intersection point of two curves as a part of the question) (20 points)



Solution:



The curves intersect each other at

$$y = x^2 - 4x + 4 = 10 - x^2 \quad \Rightarrow \quad 2\left(x^2 - 2x - 3\right) = 2\left(x+1\right)\left(x-3\right) = 0$$

$$\Rightarrow \quad \left(x = -1, y = 9\right) \;\&\; \left(x = +3, y = 1\right)$$

shown on the figure.

The area of the region asked is from x = 2 to x = 4 shown as yellow. As seen from the figure the integral should be taken from x = 2 to x = 3 and from x = 3 to x = 4 because in each part, upper and lower curves are not the same.

$$area = \int_2^3 \underbrace{\left(10 - x^2 - x^2 + 4x - 4\right)}_{-2x^2+4x+6} dx + \int_3^4 \underbrace{\left(x^2 - 4x + 4 - 10 + x^2\right)}_{2x^2-4x-6} dx =$$

$$= 2\left(-\frac{x^3}{3} + x^2 + 3x\right)\Bigg|_2^3 + 2\left(\frac{x^3}{3} - x^2 - 3x\right)\Bigg|_3^4$$

$$= 2\left\{\cancel{-9+9} + 9 + \frac{8}{3} - 4 - 6\right\} + 2\left\{\frac{64}{3} - 16 - 12 - \cancel{9+9} + 9\right\} = \frac{10}{3} + \frac{14}{3} = 8$$

4. (a) The marginal-revenue function is $dr/dq = 5,000 - 3\left(2q + 2q^2\right)$. Find the *p* demand function. (10 points)

Solution:

The revenue function, *r*, can be obtained from the marginal-revenue function:

$$r = \int \frac{dr}{dq} dq = \int \left(5,000 - 6q - 6q^2\right) dq = 5,000q - 3q^2 - 2q^3 + C .$$

Here we have to find the *C* constant of integration by assuming that when no units are sold, total revenue is zero as

$$r_{q=0} = \left(5,000q - 3q^2 - 2q^3 + C\right)_{q=0} = C = 0 .$$

We know that the general relationship of revenue in term of *p* price per unit and *q* quantity *r* = *pq*. Hence the demand function (*p* price per unit) is

$$p = \frac{r}{q} = \frac{5,000q - 3q^2 - 2q^3}{q} = 5,000 - 3q - 2q^2$$

b) The demand equation is $q = \sqrt{100 - p}$ and a supply equation of a product is $q = \frac{p}{2} - 10$. Determine the consumers' surplus (*CS*) under the market equilibrium.

Solution: (10 points)

From the demand equation we may pull the *p* price per unit as $p = 100 - q^2$ and from the supply equation $p = 2q + 20$.

First we have to find the market equilibrium where demand and supply functions are equal each other as follows:

$$f(q_0) = g(q_0)$$

$$100 - q_0^2 = 2q_0 + 20 \quad \Rightarrow \quad q_0^2 + 2q_0 - 80 = (q_0 + 10)(q_0 - 8) = 0 \quad .$$

$$\Rightarrow \quad q_0 = 8 \qquad \Rightarrow \quad p_0 = 2q_0 + 20 = 36$$

Then we can find the consumers' surplus (*CS*):

$$CS = \int_0^8 \underbrace{\left[100 - q^2 - 36\right]}_{64 - q^2} dq = \left(64q - \frac{q^3}{3}\right)_0^8 = \frac{1024}{3}$$

One may solve this problem using the integration over the *p* axis (the sample horizontal strip on the *CS* part is shown on the figure):

$$CS = \int_p f(p) dp = \int_{36}^{100} \left[\sqrt{100 - p}\right] dp, \qquad \begin{cases} u = 100 - p, & du = -dp \\ p = \{100,\ 36\} \ \Rightarrow & u = \{0,\ 64\} \end{cases}$$

$$= \int_{64}^{0} \sqrt{u}\,(-du) = \frac{2}{3} u^{3/2} \Big|_0^{64} = \frac{1024}{3}$$