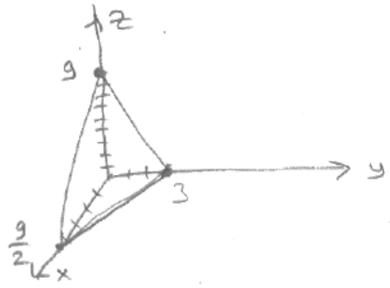
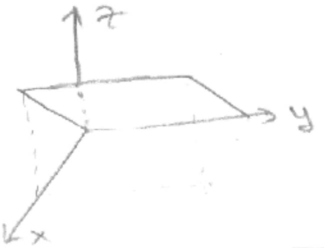
Problem Set 5 Solutions

1) a) $2x+3y+z=9$ can be put in the form $Ax+By+Cz+D=0$, so the graph is a plane. The intercepts are $(\frac{9}{2},0,0)$, $(0,3,0)$ and $(0,0,9)$



b) $z=x$ can be put in the form $Ax+By+Cz+D=0$, so the graph is a plane. Every point on the y-axis is an intercept. The x,z-trace is $z=x$ which is a line. For any fixed value of y, we obtain the line $z=x$



2) a) $\frac{\partial P}{\partial l} = 3l^2 + 0 - (1)k = 3l^2 - k$ $\qquad \frac{\partial P}{\partial k} = 0 + 3k^2 - l(1) = 3k^2 - l$

b) $\frac{\partial z}{\partial x} = \frac{(x+y)(1) - x(1)}{(x+y)^2} = \frac{y}{(x+y)^2}$ $\qquad$ because $z = x(x+y)^{-1}$ $\quad \frac{\partial z}{\partial y} = -\frac{x}{(x+y)^2}$

c) $f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

$\frac{\partial}{\partial y}[f(x,y)] = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

d) $w_x(x,y,z) = 2xyz\,e^{x^2yz}$

$w_{xy}(x,y,z) = 2xz\left[y\left(e^{x^2yz}\cdot x^2z\right) + e^{x^2yz}\cdot 1\right] = 2xz\,e^{x^2yz}(x^2yz+1)$

e) $f_x(x,y) = y\left[x\left(\frac{1}{xy}\cdot y\right) + \ln(xy)\cdot 1\right] = y\left[1+\ln(xy)\right]$

$f_{xy}(x,y) = y\left[\frac{1}{xy}\cdot x\right] + \left[1+\ln(xy)\right] = 1+1+\ln(xy) = 2+\ln(xy)$

f) $w = e^{x+y+z}\ln xyz = e^{x+y+z}(\ln x+\ln y+\ln z)$

$\frac{\partial w}{\partial y} = e^{x+y+z}(\ln x+\ln y+\ln z) + e^{x+y+z}\left(\frac{1}{y}\right) = e^{x+y+z}\left(\ln xyz + \frac{1}{y}\right)$

$w = e^{x+y+z}\ln xyz \quad \frac{\partial w}{\partial y} = e^{x+y+z}\ln xyz + e^{x+y+z}\left(\frac{xz}{xyz}\right) = e^{x+y+z}\left(\ln xyz + \frac{1}{y}\right)$

Similarly $\dfrac{\partial w}{\partial x} = e^{x+y+z}\left(\ln xyz + \dfrac{1}{x}\right)$

$$\frac{\partial^2 w}{\partial z \partial x} = e^{x+y+z}\left(\ln xyz + \frac{1}{x}\right) + e^{x+y+z}\left(\frac{1}{z}\right) = e^{x+y+z}\left(\ln xyz + \frac{1}{x} + \frac{1}{z}\right)$$

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(3) $f(x,y,z) = \dfrac{x+y}{xz} = \dfrac{1}{z} + \dfrac{y}{xz}$

$f_x(x,y,z) = -\dfrac{y}{x^2 z}$ $\qquad f_{xy}(x,y,z) = -\dfrac{1}{x^2 z}$

$f_{xyz}(x,y,z) = \dfrac{1}{x^2 z^2}$ $\qquad f_{xyz}(2,7,4) = \dfrac{1}{64}$

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(4) $f_x(x,y,z) = 6e^{y^2 \ln(z+1)}$

$f_{xy}(x,y,z) = 12y \ln(z+1)\, e^{y^2 \ln(z+1)}$

$f_{xyz}(x,y,z) = 12y\left[\ln(z+1)\left(e^{y^2 \ln(z+1)} \cdot \dfrac{y^2}{z+1}\right) + e^{y^2 \ln(z+1)} \cdot \dfrac{1}{z+1}\right]$

$f_{xyz}(0,1,0) = 12[0+1] = 12$

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(5) $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r} = (2x+2y)(e^r) + (2x+6y)\left(\dfrac{1}{r+s}\right) = 2(x+y)e^r + \dfrac{2(x+3y)}{r+s}$

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} = (2x+2y)(0) + (2x+6y)\left(\dfrac{1}{r+s}\right) = \dfrac{2(x+3y)}{r+s}$

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(6) $z^2 + \ln(yz) + \ln z + x + z = 0$ $\qquad$ or

$z^2 + \ln y + \ln z + \ln z + x + z = 0$ $\quad \Rightarrow \quad z^2 + \ln y + 2\ln z + x + z = 0$

$2z\dfrac{\partial z}{\partial y} + \dfrac{1}{y} + 2 \cdot \dfrac{1}{z}\dfrac{\partial z}{\partial y} + \dfrac{\partial z}{\partial y} = 0$

$\dfrac{\partial z}{\partial y}\left(2z + \dfrac{2}{z} + 1\right) = -\dfrac{1}{y}$ $\qquad \dfrac{\partial z}{\partial y}\left(\dfrac{2z^2 + 2 + z}{z}\right) = -\dfrac{1}{y}$

$$\frac{\partial z}{\partial y} = -\frac{z}{y(2z^2+2+z)}$$

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(7) $P = 20\, l^{0.7} k^{0.3}$ marginal productivity functions are given by

$\dfrac{\partial P}{\partial l} = 20(0.7)\, l^{-0.3} k^{0.3}$ and $\dfrac{\partial P}{\partial k} = 20(0.3)\, l^{0.7} k^{-0.7}$.

Thus $\dfrac{\partial P}{\partial l} = 14\, l^{-0.3} k^{0.3}$ and $\dfrac{\partial P}{\partial k} = 6\, l^{0.7} k^{-0.7}$

(8) $C = 3x + 0.05xy + 9y + 500$

marginal cost with respect to x is $\frac{\partial c}{\partial x} = 3 + 0.05y$

when $x=50$ and $y=100$, then $\frac{\partial c}{\partial x} = 8$

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(9) $f(x,y) = x^2 + 2y^2 - 2xy - 4y + 3$

$\left.\begin{array}{l} f_x(x,y) = 2x - 2y = 0 \\ f_y(x,y) = 4y - 2x - 4 = 0 \end{array}\right\}$ critical point $(2,2)$

$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = 4$ $\quad f_{xy}(x,y) = -2$

At $(2,2)$ $\quad D = (2)(4) - (-2)^2 = 4 > 0$ and $f_{xx}(x,y) = 2 > 0$

thus relative minimum at $(2,2)$

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(10) $\frac{\partial c}{\partial x} = 0.002(2)(x+y) + 1 = 0.004(x+y) + 1$

$\left.\frac{\partial c}{\partial x}\right|_{\substack{x=450 \\ y=550}} = 0.004(450+550) + 1 = 4 + 1 = 5$

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(11) $\frac{\partial P}{\partial L} = 40k - 6L$ $\quad \frac{\partial P}{\partial k} = 40L - 4k$

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(12) a) $\frac{\partial q_A}{\partial P_B} = \frac{\partial}{\partial P_B}\left[\frac{10 \cdot (2P_B)^{1/2}}{P_A}\right] = \frac{10\sqrt{2}}{P_A\ 2}(P_B)^{-1/2} = \frac{5\sqrt{2}}{P_A\sqrt{P_B}}$ or $\frac{10}{P_A\sqrt{2P_B}}$

b) $\frac{\partial q_B}{\partial P_A} = \frac{\partial}{\partial P_A}\left[20 + 3P_A - 2P_B\right] = 3$

c) P_A and P_B are alway positive. Since $\frac{\partial q_A}{\partial P_B} > 0$ and $\frac{\partial q_B}{\partial P_A} > 0$, A and B are competitive.

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(13) $\frac{\partial q_A}{\partial P_A} = -5$ $\quad \frac{\partial q_A}{\partial P_B} = 12P_B$ $\quad \frac{\partial q_B}{\partial P_A} = \frac{10}{P_B\sqrt{P_A}}$ $\quad \frac{\partial q_B}{\partial P_B} = -\frac{20\sqrt{P_A}}{P_B^2}$

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(14) x = constant ; z is a function of y

$\frac{\partial}{\partial y}(e^{xy} + 7x^3 + 8z - 19) = \frac{\partial}{\partial y}(0)$

$xe^{xy} + 8\frac{\partial z}{\partial y} = 0$ $\quad \frac{\partial z}{\partial y} = -\frac{xe^{xy}}{8}$

(15) y = constant ; z is a function of x

$\ln x + \ln y + \ln z + e = e^y + 1 \Rightarrow \frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}$

$\left.\frac{\partial z}{\partial x}\right|_{(e^{-2},1,e^3)} = -\frac{e^3}{e^{-2}} = -e^5$

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(16) $f_x = 8x^3y^3 - 9x^2y^3 + 4y - 1$ $\qquad f_y = 6x^4y^2 - 9x^3y^2 + 4x + 2$

$f_{xy} = 24x^3y^2 - 27x^2y^2 + 4$ $\qquad f_{yx} = 24x^3y^2 - 27x^2y^2 + 4$

$f_{yy} = 12x^4y - 18x^3y$ $\qquad f_{yx}(-1,1) = 24(-1)(1) - 27(-1)^2(1) + 4 = -47$

$f_{yyx} = 48x^3y - 54x^2y$

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(17) $\frac{\partial f}{\partial z} = -8z^3x^2e^{3y}$ $\qquad \frac{\partial^2 f}{\partial y \partial z} = -24z^3x^2e^{3y}$ $\qquad \frac{\partial^3 f}{\partial x \partial y \partial z} = -48z^3xe^{3y}$

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(18) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$

a)

$\frac{\partial z}{\partial x} = 10(x^2+y^2)^9(2x) = 20x(x^2+y^2)^9$ $\qquad \frac{\partial x}{\partial r} = 8rs^3$

$\frac{\partial z}{\partial y} = 10(x^2+y^2)^9(2y) = 20y(x^2+y^2)^9$ $\qquad \frac{\partial y}{\partial r} = 2e^{2r+3s-3}$

$\frac{\partial z}{\partial r} = 160xrs^3(x^2+y^2)^9 + 40ye^{2r+3s-3}(x^2+y^2)^9$

b) if $r=0$ and $s=1$ then $x=0,\ y=1$

$\left.\frac{\partial z}{\partial r}\right|_{\substack{r=0\\ s=1}} = 20(1)^9(2e^0) = 40$

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(19) $\left.\begin{array}{l} f_x = 2y - 3 - 2x = 0 \\ f_y = 2x - 1 - 6y = 0 \end{array}\right\}$ $\left.\begin{array}{l} -2x + 2y - 3 = 0 \\ +\ 2x - 6y - 1 = 0 \end{array}\right\}$ $-4y - 4 = 0 \Rightarrow y = -1$, $x = -5/2$

critical point is $(-\frac{5}{2}, -1)$ $\qquad D = f_{xx}f_{yy} - (f_{xy})^2$

$f_{xx} = -2 \quad f_{xy} = 2 \quad f_{yy} = -6 \qquad D = (-2)(-6) - (2)^2 = 12 - 4 = 8$

At $(-5/2, -1)$ since $D > 0$ and $f_{xx} < 0$, $f(x,y)$ has a relative maximum.

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(20) $\frac{\partial c}{\partial x} = 3x^2 - 24y = 0 \to x^2 = 8y$ $\qquad x^2 = y^4 \to y^4 = 8y \to y^3 = 8$

$\frac{\partial c}{\partial y} = 24y^2 - 24x = 0 \to y^2 = x$ $\qquad y = \sqrt[3]{8} = 2 \Rightarrow x = 4$

critical points $(0,0), (4,2)$ $\qquad \to x=0, y=0 \Rightarrow D<0$ no extrema

$\frac{\partial^2 c}{\partial x^2} = 6x$, $\frac{\partial^2 c}{\partial y^2} = 48y$, $\frac{\partial^2 c}{\partial y \partial x} = -24 \Rightarrow D = 288xy - (576)$; At $(4,2)$ $D > 0$ and $\frac{\partial^2 c}{\partial x^2} > 0$; relative min at $(4,2)$