MATH-172 ProbSet 5 Solutions

(1) a) $\begin{bmatrix} a+2d & b+2e & c+2f \\ 3a+4d & 3b+4e & 3c+4f \end{bmatrix} - \begin{bmatrix} 3d & 3e & 3f \\ 3a & 3b & 3c \end{bmatrix} = \begin{bmatrix} a-d & b-e & c-f \\ 4d & 4e & 4f \end{bmatrix}$

b) $\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 3 & 6 & 9 \\ -2 & -4 & -6 \end{bmatrix}$

size: 3×1 $\quad 1\times3 \quad 3\times3$

c) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1+6-6 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix}$

$1\times3 \quad 3\times1 \quad 1\times1$

d) $\begin{bmatrix} 3-2 & 2+4 & 1+6 \\ 9+2 & 6-4+5 & 3-6+20 \\ 15+1 & 10-2+1 & 5-3+4 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 7 \\ 11 & 7 & 17 \\ 16 & 9 & 6 \end{bmatrix}$

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(2) $c_{24} = \sum_{k=1}^{3} a_{2k} b_{k4} = a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34}$

$c_{35} = \sum_{k=1}^{3} a_{3k} b_{k5} = a_{31}b_{15} + a_{32}b_{25} + a_{33}b_{35}$

$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$

c_{53} is not available since size of C is 4×5

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(3) $a_{11} = (-1)^1(1+1) = -2$ $\qquad b_{11} = (1^2+1^2) = 2$ $\qquad b_{21} = (2^2+1^2) = 5$

$a_{12} = (-1)^1(1+2) = -3$ $\qquad b_{12} = (1^2+2^2) = 5$ $\qquad b_{22} = (2^2+2^2) = 8$

$a_{21} = (-1)^2(2+1) = 3$ $\qquad b_{13} = (1^2+3^2) = 10$ $\qquad b_{23} = (2^2+3^2) = 13$

$a_{22} = (-1)^2(2+2) = 4$

$A.B = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 & 10 \\ 5 & 8 & 13 \end{bmatrix} = \begin{bmatrix} -4-15 & -10-24 & -20-39 \\ 6+20 & 15+32 & 30+52 \end{bmatrix} = \begin{bmatrix} -19 & -34 & -59 \\ 26 & 47 & 82 \end{bmatrix}$

④ a) $3x+2y=18$
$2x-y=5$

$$\left[\begin{array}{cc|c} 3 & 2 & 18 \\ 2 & -1 & 5 \end{array}\right] \xrightarrow{\frac{R_1}{3}} \left[\begin{array}{cc|c} 1 & 2/3 & 6 \\ 2 & -1 & 5 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 2/3 & 6 \\ 0 & -7/3 & -7 \end{array}\right]$$

$$\xrightarrow{-\frac{3}{7}R_2} \left[\begin{array}{cc|c} 1 & 2/3 & 6 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{-\frac{2}{3}R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right] \qquad x=4,\ y=3$$

b) $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & -1 & -1 & -2 \\ 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow[-R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & -4 & -20 \\ 0 & 1 & -2 & -4 \end{array}\right] \xrightarrow{-\frac{R_2}{4}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & -2 & -4 \end{array}\right]$$

$$\xrightarrow[-R_2+R_1]{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow{-\frac{R_3}{3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad x=1,\ y=2,\ z=3$$

c) $x+2y+4z=6$
$x+y+2z=0$
$0x+y+2z=6$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 6 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 6 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 6 \\ 0 & -1 & -2 & -6 \\ 0 & 1 & 2 & 6 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x=-6$, $y=6-2r$, $z=r$: r is any real number

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⑤ a) $$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -1 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 4 & 0 & 0 \\ 1 & 2 & -1 \\ -3 & 1 & -1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -3 & 1 & -1 \end{bmatrix} \xrightarrow{3R_1+R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-\frac{R_2}{2}+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -1/2 \end{bmatrix} \xrightarrow[-2R_3]{-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{R_2}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

nonzero rows (3) = unknowns (3)
System must have the trivial solution ($x=y=z=0$)

b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -5 & -2 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -6 & -3 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{R_2}{2}+R_1} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

2 nonzero rows $<$ 3 unknowns.
Thus there are infinitely many solutions.

$\left.\begin{aligned} x + \tfrac{1}{2}z &= 0 \\ -2y - z &= 0 \end{aligned}\right\}$ if $z = r$

$x = -\frac{r}{2}$

$y = -r/2$

$z = r$

c) $\begin{bmatrix} 6 & 8 \\ 1 & -2 \\ 1 & 0.5 \\ 2 & 3 \end{bmatrix} \xrightarrow[\substack{-R_3+R_2 \\ -2R_3+R_4}]{-6R_3+R_1} \begin{bmatrix} 0 & 5 \\ 0 & -2.5 \\ 1 & 0.5 \\ 0 & 2 \end{bmatrix} \xrightarrow[-\frac{R_2}{2.5}]{-\frac{R_1}{5}} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0.5 \\ 0 & 1 \end{bmatrix} \xrightarrow[-\frac{R_1}{2}+R_3]{-R_1+R_2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\xrightarrow{-R_1+R_4} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 nonzero rows = 2 unknowns
The trivial solution
$x = y = 0$

6 $\left[\begin{array}{cc|cc} \frac{1}{4} & \frac{3}{8} & 1 & 0 \\ 0 & -\frac{1}{6} & 0 & 1 \end{array}\right] \xrightarrow[-6R_2]{4R_1} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 4 & 0 \\ 0 & 1 & 0 & -6 \end{array}\right] \xrightarrow{-\frac{3}{2}R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 4 & 9 \\ 0 & 1 & 0 & -6 \end{array}\right]$

The invers is $\begin{bmatrix} 4 & 9 \\ 0 & -6 \end{bmatrix}$

7 a) $(2A)^T - 3I^2 = 2A^T - 3I$

$= 2\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$

b) $A(2I) - AO^T = 2AI - AO = 2A - O = 2A = \begin{bmatrix} 2 & 2 \\ -2 & 4 \end{bmatrix}$

c) $B^3 = B^2.B = B.B.B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$

(8)

$$AX = B \qquad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$$

$$\left[\underbrace{\begin{matrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right] \xrightarrow{-R_1+R_3} \left[\begin{matrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right]$$

$$\xrightarrow[-R_2+R_3]{2R_2+R_1} \left[\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{matrix}\right] \xrightarrow[-R_3+R_2]{-2R_3+R_1} \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{matrix}}_{A^{-1}}\right]$$

$$AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -9+20-10 \\ -3+10-5 \\ 3-5+5 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{matrix} x = 1 \\ y = 2 \\ z = 3 \end{matrix}$$