MATH 172 PROBSET 2

1-A) A manufacturer’s marginal-revenue function is \( \frac{dr}{dq} = 275 - q - 0.3q^2 \)

If \( r \) is in TL, find the increase in the manufacturer’s total revenue if production is increased from 10 to 20 units.

Solution:
\[
r = \int_{10}^{20} \frac{dr}{dq} dq = \int_{10}^{20} (275 - q - 0.3q^2) dq = 275q - \frac{q^2}{2} - 0.3 \frac{q^3}{3} + C \bigg|_{10}^{20} = 1900 TL
\]

B) Evaluate the definite integrals

a) \( \int_{0}^{5} (x + x^2) dx = \frac{x^2}{2} + \frac{x^3}{3} \bigg|_{0}^{5} = \frac{25}{2} + \frac{125}{3} = \frac{325}{6} \)

b) \( \int_{5}^{10} \frac{dx}{x-1} = \ln |x-1| \bigg|_{2}^{10} = \ln 9 - \ln 1 = 2 \ln 3 \)

c) \( \int_{0}^{1} \frac{x^2 + x + \sqrt{x+1}}{x+1} dx = \int_{0}^{1} \left( \frac{x^2 + x + \sqrt{x+1}}{x+1} \right) dx = \int_{0}^{1} \left( x + (x+1)^{\frac{1}{2}} \right) dx = \frac{x^2}{2} + 2(x+1)^{\frac{3}{2}} \bigg|_{0}^{1} = -\frac{3}{2} + 2\sqrt{2} \)

d) \( \int_{0}^{2} x^2 e^x dx = \left[ \frac{x^2 e^x}{3} + \frac{1}{3} e^x \right]_{0}^{2} = \frac{1}{3} (e^2 - 1) \)

\[\begin{align*}
&u = x^3 \\
&du = 3x^2 dx:
\end{align*}\]

\[\begin{align*}
&x = 2 \Rightarrow u = 2^3 \\
&x = 0 \Rightarrow u = 0
\end{align*}\]

e) \( \int_{\sqrt{3}}^{2} 7x\sqrt{4-x^2} dx = \int_{0}^{\frac{\pi}{2}} 7 \sqrt{u} \frac{du}{-2} = \frac{7}{2} \int_{0}^{\frac{\pi}{2}} \sqrt{u} du = \frac{7}{2} \left( \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right)_{0}^{\frac{\pi}{2}} = \frac{7}{3} \)

\[\begin{align*}
&u = 4-x^2 \\
&du = -2x dx:
\end{align*}\]

\[\begin{align*}
&x = 2 \Rightarrow u = 0 \\
&x = \sqrt{3} \Rightarrow u = 1
\end{align*}\]

C) If \( \int_{1}^{5} f(x) dx = 6 \) and \( \int_{1}^{3} f(x) dx = 2 \), find \( \int_{1}^{3} f(x) dx \).

Solution:
\[
\int_{1}^{3} f(x) dx = \int_{1}^{5} f(x) dx - \int_{3}^{5} f(x) dx = \int_{1}^{5} f(x) dx + \int_{5}^{3} f(x) dx = 6 + 2 = 8
\]

2) Find the area of the region bounded by the curve, lines and x-axis. Sketch the region on the x-y plane.

a) \( y = x^2 - 1, \ x = 0, \ x = 2 \)

Solution:

The curve \( y = x^2 - 1 \) is just a parabola whose intercepts are given by
\[\begin{align*}
&x = 0 \quad \Rightarrow \quad y = 0^2 - 1 = -1 \\
&y = 0 \quad \Rightarrow \quad 0 = x^2 - 1 = (x-1)(x+1) \quad x = 1 \ or \ -1
\end{align*}\]
The area can be taken by splitting the integral from 0 to 1 and from 1 to 2 because the area from 0 to 1 is under the x-axis and the negative so we have to multiply by “-” sign to make it positive.

\[ \text{Area} = -\int_0^1 (x^2 - 1) \, dx + \int_1^2 (x^2 - 1) \, dx = \left[ -\frac{x^3}{3} + x \right]_0^1 + \left[ \frac{x^3}{3} - x \right]_1^2 = -\frac{1}{3} + 1 + \frac{8}{3} - 2 - \frac{1}{3} + 1 = 2 \]

b) \( y = x^2 - 1, \ y = 0 \)

Solution:

The area can be taken from -1 to 1, but it is under the x-axis and the negative so we have to multiply by “-” sign to make it positive.

\[ \text{Area} = -\int_{-1}^1 (x^2 - 1) \, dx = \left[ -\frac{x^3}{3} + x \right]_{-1}^1 = -\frac{1}{3} + 1 - \frac{1}{3} + 1 = \frac{4}{3} \]

c) \( y = x^2 + 2x - 3, \ x = -1, \ x = 2 \)

Solution:

The curve \( y = x^2 + 2x - 3 \) is just a parabola whose intercepts are given by

\[ x = 0 \quad \Rightarrow \quad y = -3 \]

\[ y = 0 \quad \Rightarrow \quad 0 = x^2 + 2x - 3 = (x+3)(x-1) \quad \Rightarrow \quad x = -3 \text{ or } 1 \]
The area can be taken by splitting the integral from -1 to 1 and from 1 to 2 because the area from -1 to 1 is under the x-axis and the negative so we have to multiply by "-" sign to make it positive.

\[
\begin{align*}
\text{Area} &= -\int_{-1}^{1} (x^2 + 2x - 3) \, dx + \int_{1}^{2} (x^2 + 2x - 3) \, dx \\
&= \left[ -\frac{x^3}{3} - x^2 + 3x \right]_{-1}^{1} + \left[ \frac{x^3}{3} + x^2 - 3x \right]_{1}^{2} \\
&= \frac{16}{3} - \frac{7}{3} = \frac{23}{3}
\end{align*}
\]

d) Evaluate \( \int_{-1}^{2} (x^2 + 2x - 3) \, dx \) and compare the results with the results of c

Solution:

\[
\int_{-1}^{2} (x^2 + 2x - 3) \, dx = \left[ \frac{x^3}{3} + x^2 - 3x \right]_{-1}^{2} = -3
\]

We obtained the negative result which cannot be area from -1 to 2. Actually in this case, negative part from -1 to 1 is kept as negative so the area below the x-axis is subtracted from the area above x-axis from 1 to 2 such as \(- \frac{16}{3} + \frac{7}{3} = -3\).

e) \quad y = \frac{1}{x - 2}, \quad x = 3, \quad x = e^2 + 2

Solution:

\[
\begin{align*}
\text{Area} &= \int_{3}^{e^2 + 2} \frac{1}{x - 2} \, dx = \ln(x - 2) \bigg|_{3}^{e^2 + 2} = \ln e^2 - \ln 1 = 2
\end{align*}
\]
f) \( y = e^x, \ x = -1, \ x = 1 \)

Solution:

\[
\text{Area} = \int_{-1}^{1} e^x \, dx = e^x \bigg|_{-1}^{1} = e - e^{-1}
\]

\[g) \quad y = \frac{1}{2}(e^x + e^{-x}), \ x = -1, \ x = 1 \]

Solution:

\[
\text{Area} = \frac{1}{2} \int_{-1}^{1} \left( e^x + e^{-x} \right) \, dx = \left[ \frac{1}{2}(e^x - e^{-x}) \right]_{-1}^{1} = \frac{1}{2} \left( e - e^{-1} - e^{-1} + e \right) = e - e^{-1}
\]

3) Find the area of the region bounded by the given curves and lines. Sketch the region on the x-y plane.

a) \( y = x^2, \ y = -x^2 + 2 \)

Solution:

The intercepts of the two plots are:

\[ x^2 = -x^2 + 2 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1 \]
\[ \text{Area} = \int_{-1}^{1} \left( 2 - x^2 - x^3 \right) dx = 2x - \frac{2}{3}x^3 \bigg|_{-1}^{1} = e - e^{-1} \]

b) \( y = x^2, \ y = x \)

Solution:

The intercepts of the two plots are:

\[ x^2 = x \quad \Rightarrow \quad x(x-1) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 1 \]

\[ \text{Area} = \int_{0}^{1} \left( x - x^2 \right) dx = \frac{x^2}{2} - \frac{x^3}{3} \bigg|_{0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \]

c) \( y = x^2, \ y = x, \ x = 3 \)

Solution:
\[
Area = \int_0 ^1 (x-x^3) \, dx + \int_1 ^3 (x^2-x) \, dx = \frac{x^2}{2} - \frac{x^3}{3} \bigg|_0 ^1 + \frac{x^3}{3} - \frac{x^2}{2} \bigg|_1 ^3 = \frac{1}{6} + \frac{14}{3} = \frac{29}{6}
\]

d) \( y = -3x^2 + 3, \ y = 3x + 3 \)

Solution:
The intercepts of the two plots are:
\(-3x^2 + 3 = 3x + 3 \Rightarrow 3x(x+1) = 0 \Rightarrow x = 0 \) or \(-1\)

\[
\int_{-1}^{1} (-3x^2 + 3x - 3x - 3) \, dx = -\frac{3x^3}{3} - \frac{3x^2}{2} \bigg|_{-1}^{1} = -1 + \frac{3}{2} = \frac{1}{2}
\]

e) \( y = -3x^2 + 3, \ y = 3x + 3, \ y = 1 \)

Solution:
The intercepts of the line \( y = 1 \) with \( y = -3x^2 + 3 \) and \( y = 3x + 3 \) are:
\( y = -3x^2 + 3 = 1 \) \( \Rightarrow \) \( x = \pm \sqrt{\frac{2}{3}} \)
\( y = 3x + 3 = 1 \) \( \Rightarrow \) \( x = -\frac{2}{3} \)

The darker area on the plot is the area to be found.
The area can be calculated from $x = -1$ to $x = -\sqrt{2/3}$ and $x = -\sqrt{2/3}$ to $x = -2/3$. So

$$\begin{aligned} \text{Area} &= \int_{-1}^{-\sqrt{2/3}} (-3x^2 + \beta - 3x - \beta) \, dx + \int_{-\sqrt{2/3}}^{-2/3} (1 - 3x - 3) \, dx \\
&= \left[ -\frac{\beta x^3}{3} - \frac{3x^2}{2} \right]_{-1}^{-\sqrt{2/3}} + \left[ -\frac{3x^2}{2} - 2x \right]_{-\sqrt{2/3}}^{-2/3} = 0.05 + 2.63 = 2.68 \end{aligned}$$

4) Express the area of the region bounded by the given curves and lines in terms of definite integral or integrals.

a) $y = (x - 1)^2$, $y = x + 5$

Solution:
The intercepts of the two plots are:

$y = (x - 1)^2 = x + 5 \quad \Rightarrow \quad (x+1)(x-4) = 0 \quad \Rightarrow \quad x = -1$ or $4$

$$\begin{aligned} \text{Area} &= \int_{-1}^{4} ((x+5) - (x - 1)^2) \, dx = \int_{-1}^{4} (-x^2 + 3x + 4) \, dx = \frac{43}{2} \end{aligned}$$

b) $y = x^2 + 2x - 1$, $y = 2$

Solution:
The intercepts of the two plots are:

$y = x^2 + 2x - 1 = 2 \quad \Rightarrow \quad (x+3)(x-1) = 0 \quad \Rightarrow \quad x = -3$ or $1$

$$\begin{aligned} \text{Area} &= \int_{-3}^{1} (2 - x^2 - 2x + 1) \, dx = \int_{-3}^{1} (-x^2 - 2x + 3) \, dx = -\frac{x^3}{3} - x^2 + 3x \bigg|_{-3}^{1} = \frac{32}{3} \end{aligned}$$
5) Profit of a company is a function of units sold \( q \) and is given by the following function: 
\[
f(q) = 4 - q - \frac{1}{q^2}
\]
\( q \) changes between 10 and 30. Evaluate the average profit per unit using the integral:
\[
I = \frac{1}{20} \int_{10}^{30} f(q) dq
\]
\[\text{answer: } -16 - \frac{\ln 3}{20}\]

Solution:
\[
I = \frac{1}{20} \int_{10}^{30} f(q) dq = \frac{1}{20} \int_{10}^{30} \left(4 - q - \frac{1}{q^2}\right) dq = \frac{1}{20} \left(4q - \frac{q^2}{2} - \ln q\right)_{10}^{30} = -16 - \frac{\ln 3}{20}
\]

6) Demand and supply equations are given respectively. Determine consumer and producer surpluses under market equilibrium.

a) \( p = 100 - q^2, \ p = 2q + 20 \)

Solution:
Equilibrium point is
\[
p = 100 - q^2 = 2q + 20 \quad \Rightarrow \quad q^2 + 2q - 80 = 0 = (q + 10)(q - 8) \quad \Rightarrow \quad q > -10 \quad \text{or} \quad q_0 = 8
\]
\[
p_0 = 2q_0 + 20\bigg|_{q_0=8} = 36
\]
\[
CS = \int_0^{q_0} (f(q) - p_0) dq = \int_0^{8} \left(100 - q^2 - 36\right) dq = 64q - \frac{q^3}{3}\bigg|_0^8 = \frac{1024}{3}
\]
\[
PS = \int_0^{q_0} (p_0 - g(q)) dq = \int_0^{8} \left(36 - 2q - 20\right) dq = 16q - q^2\bigg|_0^8 = 64
\]

b) \( p = 1500 - q^2, \ p = 700 + q^2 \)

Solution:
Equilibrium point is
\[
p = 1500 - q^2 = 700 + q^2 \quad \Rightarrow \quad 2q^2 = 800 \quad \Rightarrow \quad q_0 = \sqrt{400} = 20
\]
\[
p_0 = 700 + q_0^2\bigg|_{q_0=20} = 1100
\]
\[
CS = \int_0^{q_0} (f(q) - p_0) dq = \int_0^{20} \left(1500 - q^2 - 1100\right) dq = 400q - \frac{q^3}{3}\bigg|_0^{20} = \frac{16000}{3}
\]
\[
PS = \int_0^{q_0} (p_0 - g(q)) dq = \int_0^{20} \left(1100 - 700 - q^2\right) dq = 400q - \frac{q^3}{3}\bigg|_0^{20} = \frac{16000}{3}
\]
7) Marginal cost function of a product is given; a) Determine the marginal cost when 90 units are produced, b) If fixed cost is $500, find the total cost of producing 90 units.

\[
\frac{dc}{dq} = 10 - \frac{100}{q + 10}
\]

Solution:

\[c = \int \frac{dc}{dq} dq = \int \left(10 - \frac{100}{q + 10}\right) dq = 10q - 100\ln(q + 10) + C\]

When \(q = 0\),

\[c = 500 = 10q - 100\ln(q + 10) + C |_{q=0} = -100\ln10 + C \Rightarrow C = 500 + 100\ln10\]

Hence the cost function is given by

\[c = 10q - 100\ln(q + 10) + 500 + 100\ln10 = 10q - 100\ln\left(\frac{q}{10} + 1\right) + 500\]

To find the total cost of producing 90 units:

\[c = 10q - 100\ln\left(\frac{q}{10} + 1\right) + 500 |_{q=90} = 900 - 100\ln10 + 500 = 1400 - 100\ln10 \approx \$1170\]

8) The demand equation for a product is \( p = 0.01q^2 - 1.1q + 30 \) and the supply equation is \( p = 0.01q^2 + 8 \). Determine consumers' surplus and producers' surplus when market equilibrium has been established.

Solution:

Equilibrium point is \( p = 0.01q^2 - 1.1q + 30 = 0.01q^2 + 8 \) \( \Rightarrow q_0 = \frac{22}{1.1} = 20 \)

\[p_0 = 0.01q^2 + 8 |_{q=20} = 12\]

Consumers' surplus:

\[CS = \int_{0}^{q_0} (f(q) - p_0) dq = \int_{0}^{20} (0.01q^2 - 1.1q + 30 - 12) dq = 0.01\frac{q^3}{3} - 1.1\frac{q^2}{2} + 18q |_{0}^{20} = \frac{500}{3}\]

Producers' surplus:

\[PS = \int_{0}^{q_0} (p_0 - g(q)) dq = \int_{0}^{20} (12 - 0.01q^2 - 8) dq = 4q - 0.01\frac{q^3}{3} |_{0}^{20} = \frac{160}{3}\]