1. A company's production function is given by \( P = 2L^2 - 3k^2 + 10Lk - \frac{500}{k} \), where \( P \) is the total output generated by \( L \) units of labor and \( k \) units of capital. Determine the marginal production function with respect to \( L \) when \( k = 50 \) units and \( L = 10 \) units. (10 Points)

**Solution:**

\[
\frac{\partial P}{\partial L} = 4L + 10k
\]

\[
\left. \frac{\partial P}{\partial L} \right|_{L=10, k=50} = 4L + 10k = 40 + 500 = 540
\]

2. Evaluate the integral \( \int \frac{x+1}{e^x} \, dx \). (10 Points)

**Solution:**

\[
\int \frac{x+1}{e^x} \, dx = - (x+1) e^{-x} - \int -e^{-x} \, dx = -(x+1) e^{-x} - e^{-x} + C
\]

\[
= -(x+2) e^{-x} + C
\]
3. Let \( q_A = 50 - 5p_A + 6p_B^2 \) and \( q_B = 20\sqrt{p_A^{-1}p_B^{-1}} \) be demand functions, where \( p_A \) and \( p_B \) are prices for products A and B, respectively. Determine: (i) the marginal demand for A with respect to \( p_B \), (ii) the marginal demand for B with respect to \( p_A \), (iii) whether A and B are competitive, complementary, or neither. (10 Points)

Solution:

\[
\begin{align*}
(i) \quad \frac{\partial q_A}{\partial p_B} &= 12p_B > 0 \\
(ii) \quad \frac{\partial q_B}{\partial p_A} &= 10p_A^{-3/2}p_B^{-1} > 0
\end{align*}
\]

(iii) From (i) and (ii), the products A and B are competitive products.
4. Let \( \ln\sqrt{y(z+y)} - \frac{1}{2}x = z \). Use implicit differentiation to evaluate \( \frac{\partial z}{\partial x} \). (10 Points)

Solution:
\[
\ln\sqrt{y(z+y)} - \frac{1}{2}x = z
\]
\[
\frac{1}{2}\ln y + \frac{1}{2}\ln(z+y) - \frac{1}{2}x = z \quad \Rightarrow \quad \ln y + \ln(z+y) - x = 2z
\]
\[
\frac{\partial}{\partial x} (\ln y + \ln(z+y) - x = 2z)
\]
\[
\frac{\partial}{\partial x} \ln(z+y) - \frac{\partial x}{\partial x} = 2 \frac{\partial z}{\partial x}
\]
\[
\frac{\partial z}{\partial x} \left( \frac{1}{z+y} - 2 \right) = 1 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{1}{z+y - 2} = \frac{z+y}{1-2(z+y)}
\]

5. If \( w = e^{x-y} + x^2 - y^2 \) where \( x = rs \), \( y = s^2 + r^2 \), evaluate \( \frac{\partial w}{\partial r} \) when \( r = 1 \) and \( s = 0 \). (10 Points)

Solution:
\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}
\]
\[
= (e^{x-y} + 2x)s + (-e^{x-y} - 2y)2r
\]
\[
= (s - 2r)e^{x-y} + 2xs - 4yr
\]
when \( r = 1 \) and \( s = 0 \), \( x = 0 \) and \( y = 1 \).
\[
\left. \frac{\partial w}{\partial r} \right|_{r=1, s=0} = \left( (s - 2r)e^{x-y} + 2xs - 4yr \right)_{r=1, s=0}
\]
\[
= -2e^{-1} - 4
\]
6. Examine the function \( f(x, y) = x^3 + 3y^2 - 3xy + 7 \) for relative extrema using the second derivative test. (15 Points)

**Solution:**
The first derivatives of \( f(x, y) \) function:

\[
\begin{align*}
    f_x &= \frac{\partial f}{\partial x} = 3x^2 - 3y = 0 \quad \Rightarrow \quad y = x^2 \\
    f_y &= \frac{\partial f}{\partial y} = 6y - 3x = 6x^2 - 3x = 3x(2x - 1) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = \frac{1}{2}
\end{align*}
\]

The critical points are at (0,0) and (1/2, 1/4). The second derivatives are as follows

\[
\begin{align*}
    f_{xx} &= 6x, \quad f_{xy} = -3, \quad f_{yy} = 6
\end{align*}
\]

The function \( D(x, y) \) for second-derivative test is given by

\[
D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36x - 9 = 9(4x - 1)
\]

The (0, 0) point: \( D(0,0) = 9(4 \times 0 - 1) = -9 < 0 \), saddle point

The (1/2, 1/4) point: \( D(\frac{1}{2}, \frac{1}{4}) = 9\left(4 \times \frac{1}{2} - 1\right) = 9 > 0 \), \( f_{xx} = 0 > 0 \) relative minimum
7. By using matrix reduction, solve the given system and determine whether it has unique solution or infinitely many solutions. (15 Points)

\[
\begin{align*}
2y + z &= 0 \\
x - 2z &= 0 \\
-y + z &= 0
\end{align*}
\]

Solution:

\[
\begin{bmatrix}
0 & 2 & 1 \\
1 & 0 & -2 \\
0 & -1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -2 \\
0 & 2 & 1 \\
0 & -1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -2 \\
0 & -1 & 1 \\
0 & 2 & 1
\end{bmatrix}
\]

The number of equation is the same as the unknowns; therefore it has unique solutions as \(x = 0, \ y = 0, \ z = 0\).
8. Solve the given system by using the inverse of its coefficient matrix. (20 Points)

\[
\begin{align*}
  x - 3y &= -4 \\
  x - y &= 2
\end{align*}
\]

**Solution:**

The equation can be written in matrix form as:

\[
AX = B \quad \text{with} \quad A = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 2 \end{bmatrix}
\]

The inverse of the coefficient matrix \( A \) can be found as follows:

\[
\begin{bmatrix} A | I \end{bmatrix} = 
\begin{bmatrix} 1 & -3 | 1 & 0 \\ 1 & -1 | 0 & 1 \end{bmatrix} 
\overset{-R_1 + R_2}{\rightarrow} 
\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} 
\overset{-\frac{3}{2}R_2}{\rightarrow} 
\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}
\]

Hence we may solve the system of equations using the inverse of \( A \) matrix by

\[
X = A^{-1}B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}
\]