1. Evaluate the following integrals. (10 points each)

(a) \( \int (4 - 3z)^4 \, dz \)

Solution:
\[
\int (4 - 3z)^4 \, dz = \int u^4 \, \frac{du}{-3} = -\frac{1}{3} \frac{u^5}{5} + C = -\frac{1}{15} (4 - 3z)^3 + C \quad \left\{ u = 4 - 3z \right\} \quad \left\{ du = -3 \, dz \right\}
\]

(b) \( \int 6x^2 e^{x^3 - 2} \, dx \)

Solution:
\[
\int 6x^2 e^{x^3 - 2} \, dx = \int 2e^u \, du = 2e^u + C = 2e^{x^3 - 2} + C \quad \left\{ u = x^3 - 2 \right\} \quad \left\{ du = 3x^2 \, dx \right\}
\]

(c) \( \int \left( \frac{1 - 2x^5}{x^3} \right) \, dx \)

Solution:
\[
\int \left( \frac{1 - 2x^5}{x^3} \right) \, dx = \int \left( \frac{1}{x^3} - 2x^2 \right) \, dx = \int x^{-3} \, dx - 2 \int x^2 \, dx = -\frac{1}{2x^2} - \frac{2}{3} x^3 + C
\]
2. Find the following integrals (10 points each)

(a) \[ \int_0^1 \frac{x + 1}{e^x} \, dx \]

Solution:
\[ \int_0^1 \frac{x + 1}{e^x} \, dx = -(x + 1)e^{-x} \bigg|_0^1 - \int_0^1 e^{-x} \, dx \quad \Rightarrow \quad u = x + 1 \quad \Rightarrow \quad du = dx \]
\[ dv = e^{-x} \, dx \quad \Rightarrow \quad v = -e^{-x} \]
\[ = -2e^{-1} + 1 - e^{-0} = 2 - 3e^{-1} \]

(b) \[ \int \frac{\ln(x + 1)}{x + 1} \, dx \]

Solution:
\[ \int (x + 1)\ln(x + 1) \, dx = \frac{1}{2} (x + 1)^2 \ln(x + 1) - \frac{1}{2} (x + 1)^2 \frac{1}{x + 1} \, dx \]
\[ = \frac{1}{2} (x + 1)^2 \ln(x + 1) - \frac{1}{4} (x + 1)^2 + C \]
\[ = \frac{1}{4} (x + 1)^2 (2\ln(x + 1) - 1) + C \]

(c) \[ \int \frac{5x^2 + 2}{x^3 + x} \, dx \]

Solution:
First we have to express the rational function in terms of partial fraction:
\[ \frac{5x^2 + 2}{x^3 + x} = \frac{5x^2 + 2}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{Ax^2 + A + Bx^2 + Cx}{x^3 + x} \]
\[ 5x^2 + 2 = (A + B)x^2 + Cx + A \]
\[ C = 0; \quad A = 2; \quad A + B = 5 \quad \Rightarrow \quad B = 3 \]
\[ \int \frac{5x^2 + 2}{x^3 + x} \, dx = \int \frac{2}{x} \, dx + \int \frac{3x}{x^2 + 1} \, dx = 2\ln|x| + 3\int u^{-1} \, du \]
\[ = 2\ln|x| + \frac{3}{2} \ln(x^2 + 1) + C = \ln \left( x^2(x^2 + 1)^{3/2} \right) + C \]
3. Express the area of the shaded region in terms of an integral (or integrals) and evaluate. (You have to find the intersection point of two curves as a part of the question) (20 points)

Solution:

The curves intersect each other at

\[ y = x^2 - 4x + 4 = 10 - x^2 \quad \Rightarrow \quad 2(x^2 - 2x - 3) = 2(x + 1)(x - 3) = 0 \]

\[ \Rightarrow \quad (x = -1, y = 9) \quad \& \quad (x = +3, y = 1) \]

shown on the figure.

The area of the region asked is from \( x = 2 \) to \( x = 4 \) shown as yellow. As seen from the figure the integral should be taken from \( x = 2 \) to \( x = 3 \) and from \( x = 3 \) to \( x = 4 \) because in each part, upper and lower curves are not the same.

\[
\text{area} = \int_2^3 (10 - x^2 - x^2 + 4x - 4)dx + \int_3^4 (x^2 - 4x + 4 - 10 + x^2)dx = \\
= 2\left(-\frac{x^3}{3} + x^2 + 3x\right)_2^3 + 2\left(\frac{x^3}{3} - x^2 - 3x\right)_3^4 \\
= 2\left(-\frac{9}{3} + 9 + \frac{8}{3} - 4 - 6\right) + 2\left(\frac{64}{3} - 16 - 12 - 9\right) = 10 + 14 = 8
\]
4. (a) The marginal-revenue function is \( dr/dq = 5,000 - 3(2q + 2q^2) \). Find the \( p \) demand function. (10 points)

Solution:
The revenue function, \( r \), can be obtained from the marginal-revenue function:
\[
r = \int dr/dq dq = \int \left(5,000 - 6q - 6q^2\right) dq = 5,000q - 3q^2 - 2q^3 + C.
\]
Here we have to find the \( C \) constant of integration by assuming that when no units are sold, total revenue is zero as
\[
r_{q=0} = \left(5,000q - 3q^2 - 2q^3 + C\right)_{q=0} = C = 0.
\]
We know that the general relationship of revenue in term of \( p \) price per unit and \( q \) quantity \( r = pq \). Hence the demand function (\( p \) price per unit) is
\[
p = \frac{r}{q} = \frac{5,000q - 3q^2 - 2q^3}{q} = 5,000 - 3q - 2q^2.
\]

b) The demand equation is \( q = \sqrt{100 - p} \) and a supply equation of a product is \( q = \frac{p}{2} - 10 \). Determine the consumers’ surplus (CS) under the market equilibrium.

Solution: (10 points)

From the demand equation we may pull the \( p \) price per unit as \( p = 100 - q^2 \) and from the supply equation \( p = 2q + 20 \).
First we have to find the market equilibrium where demand and supply functions are equal each other as follows:

\[
f(q_0) = g(q_0)
\]
\[
100 - q_0^2 = 2q_0 + 20 \quad \Rightarrow \quad q_0^2 + 2q_0 - 80 = (q_0 + 10)(q_0 - 8) = 0.
\]
\[
\Rightarrow \quad q_0 = 8 \quad \Rightarrow \quad p_0 = 2q_0 + 20 = 36
\]
Then we can find the consumers’ surplus (CS):
\[
CS = \int_0^8 \left[100 - q^2 - 36\right] dq = \left(64q - \frac{q^3}{3}\right)_0^8 = \frac{1024}{3}
\]
One may solve this problem using the integration over the \( p \) axis (the sample horizontal strip on the CS part is shown on the figure):
\[
CS = \int_p f(p)dp = \int_{36}^{100} \sqrt{100 - p} dp, \quad \left\{ \begin{array}{l}
u = 100 - p, \quad du = -dp \\
p = \{100, 36\} \quad \Rightarrow \quad u = \{0, 64\}
\end{array} \right.
\]
\[
= \int_{64}^{0} \sqrt{u} (-du) = \frac{2}{3} u^{3/2}\bigg|_0^{64} = \frac{1024}{3}
\]