Data Structures – Week #6

Special Trees

Outline

- Adelson-Velskii-Landis (AVL) Trees
- Splay Trees
- B-Trees

AVL Trees

Motivation for AVL Trees

- Accessing a node in a BST takes *$O(\log_2 n)$* in average.
- A BST can be structured so as to have an average access time of *$O(n)$*. *Can you think of one such BST?*
- Q: Is there a way to *guarantee a worst-case access time of $O(\log_2 n)$ per node* or can we find a way to *guarantee a BST depth of $O(\log_2 n)$*?
- A: ***AVL Trees***

Definition

An *AVL tree* is a *BST* with the following *balance condition*:

for each node in the BST, the height of left and right sub-trees can differ by at most 1, or

$$\left| h_{N_L} - h_{N_R} \right| \leq 1.$$

Remarks on Balance Condition

- *Balance condition must be easy to maintain*:
  - This is the reason, for example, for the balance condition's not being as follows: the height of left and right sub-trees of each node have the same height.
- *It ensures the depth of the BST is $O(\log_2 n)$.*
- The *height information is stored* as an additional field in BTNodeType.

Structure of an AVL Tree

```
struct BTNodeType {
    infoType *data;
   unsigned int height;
    struct BTNodeType *left;
    struct BTNodeType *right;
 }
```

Rotations

Definition:

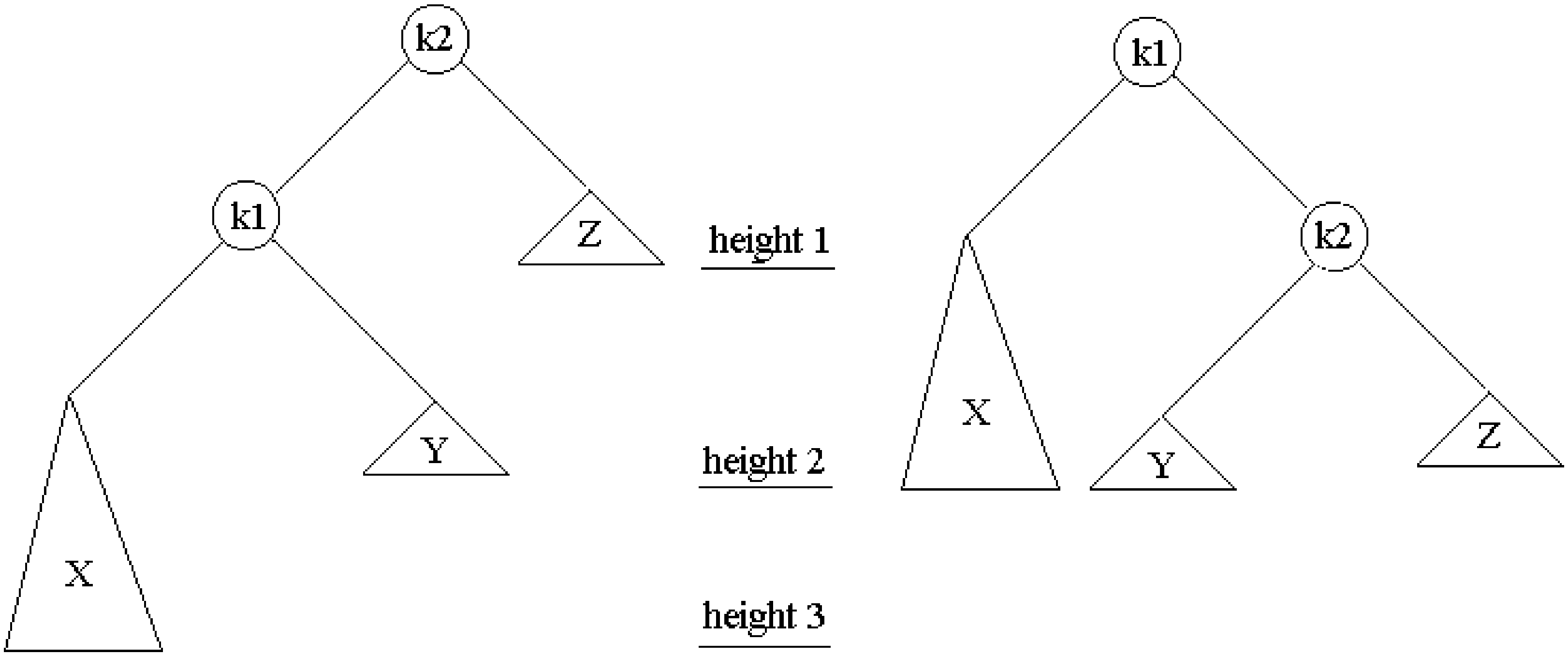
- *Rotation* is the operation performed on a BST to restore its AVL property lost as a result of an insert operation.

- We consider the node α whose new balance violates the AVL condition.

Rotation

- Violation of AVL condition
- The AVL condition violation may occur in four cases:
  - Insertion into *left subtree of the left child (L/L)*
  - Insertion into *right subtree of the left child (R/L)*
  - Insertion into *left subtree of the right child (L/R)*
  - Insertion into *right subtree of the right child (R/R)*
- The outside cases 1 and 4 (i.e., L/L and R/R) are fixed by a *single rotation*.
- The other cases (i.e., R/L and L/R) need two rotations called *double rotation* to get fixed.
- These are fundamental operations in balanced-tree algorithms.

Single Rotation (L/L)

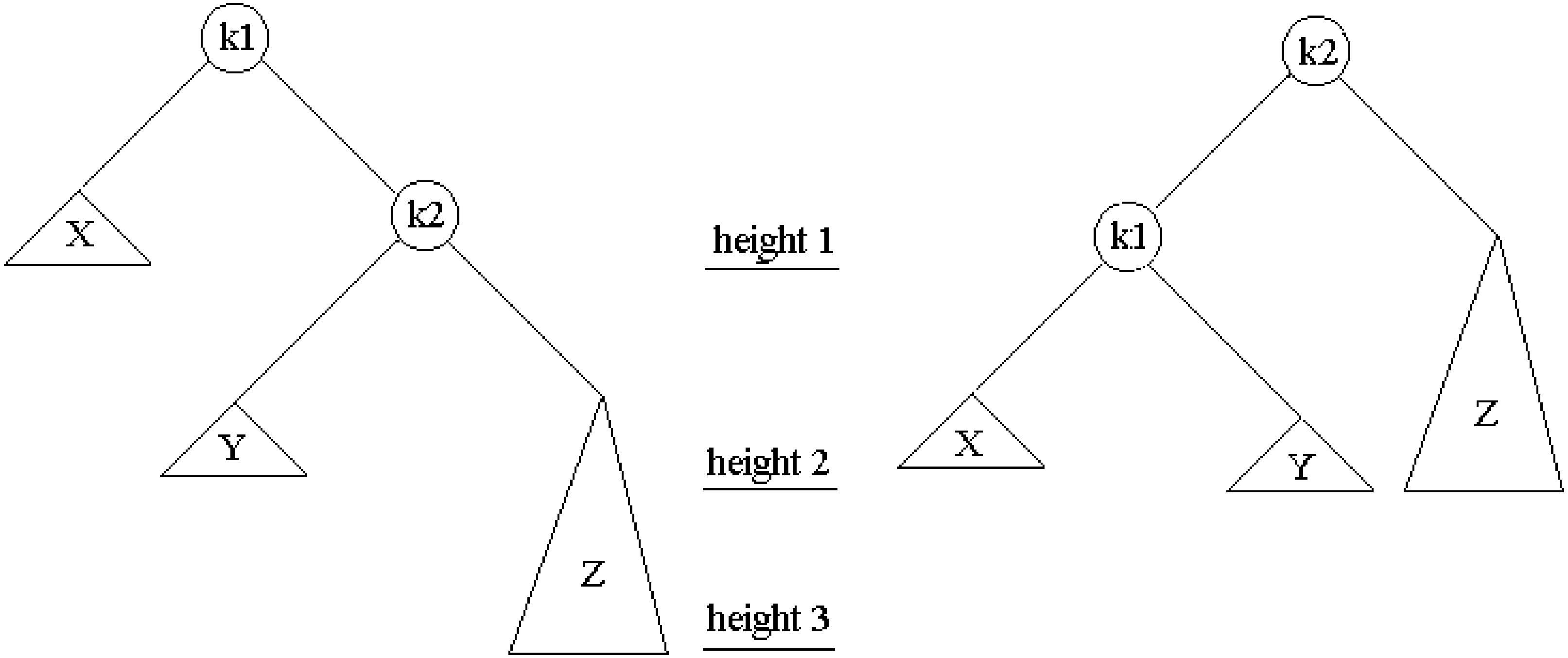


before single rotation

after single rotation

$\alpha \equiv$ k2 node

Single Rotation (R/R)



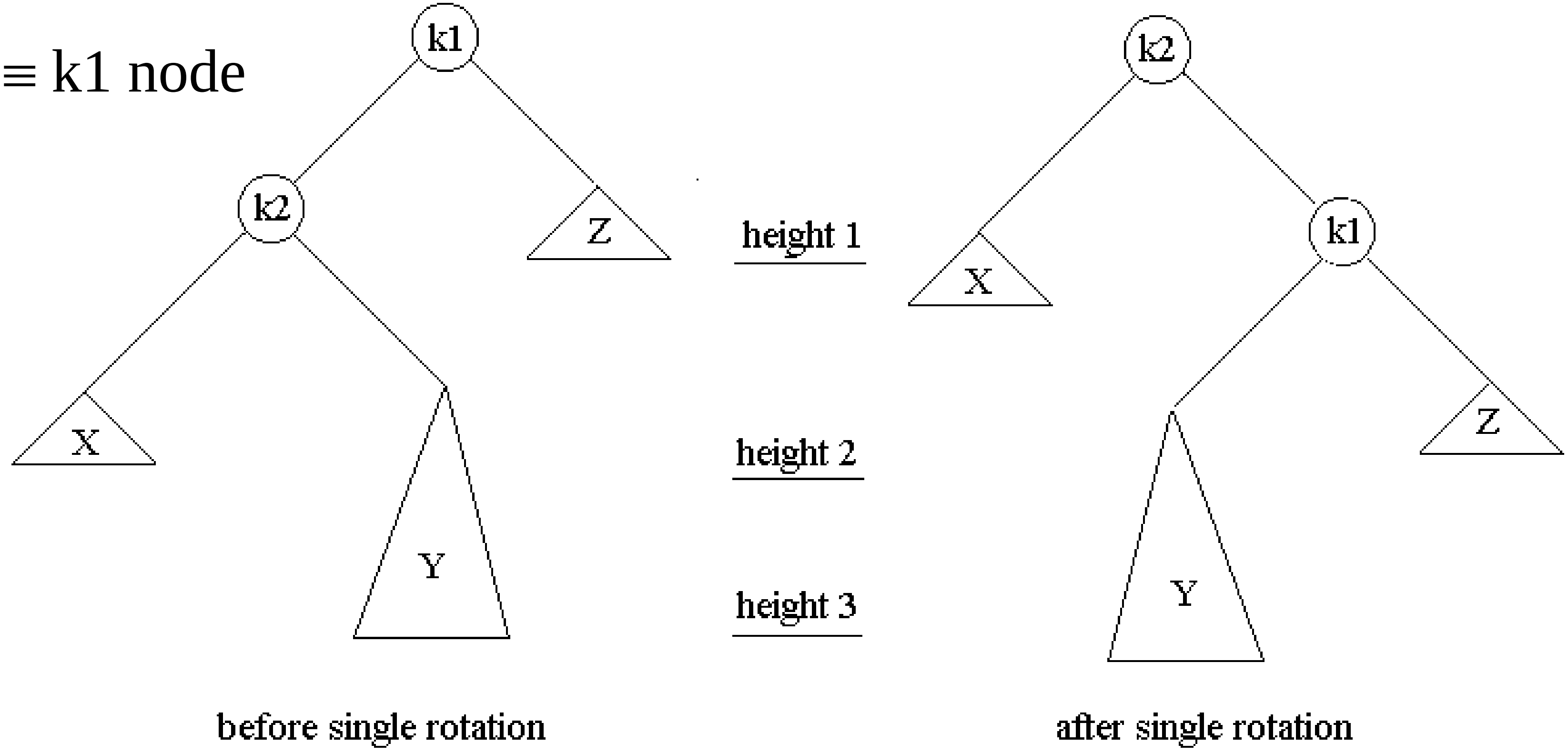
before single rotation

after single rotation

$\alpha \equiv$ k1 node

Double Rotation (R/L)

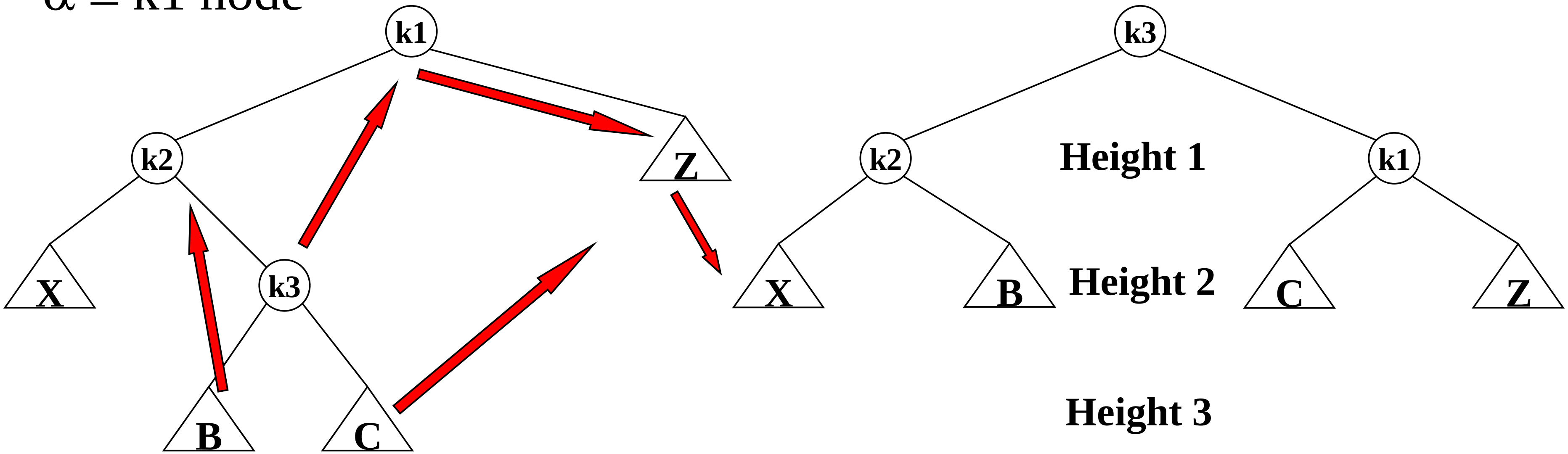
α ≡ k1 node



Single rotation cannot fix the AVL condition violation!!!

Double Rotation (R/L)

α ≡ k1 node



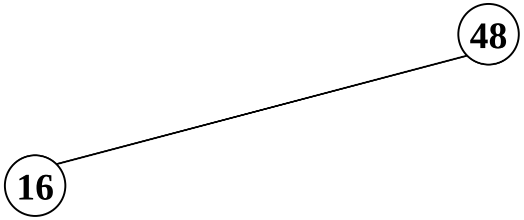
The symmetric case (L/R) is handled similarly left as an exercise to you!

Constructing an AVL Tree – Animation

48

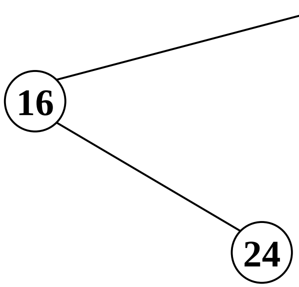
Constructing an AVL Tree – Animation

48 16



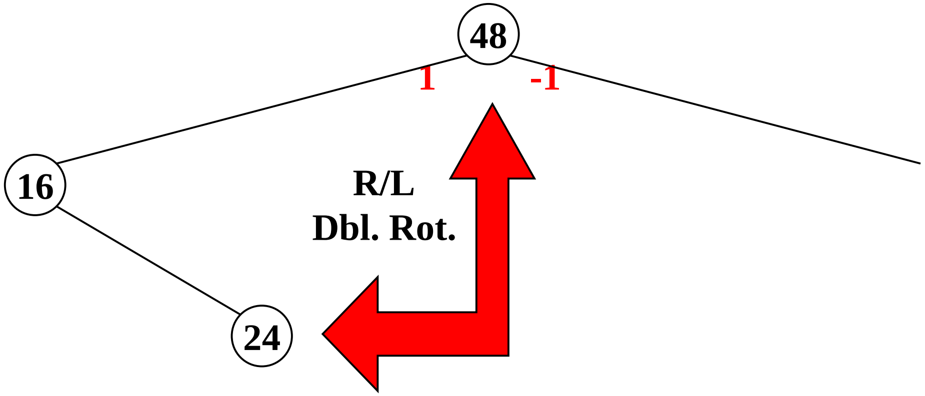
Constructing an AVL Tree – Animation

48 16 24



Constructing an AVL Tree – Animation

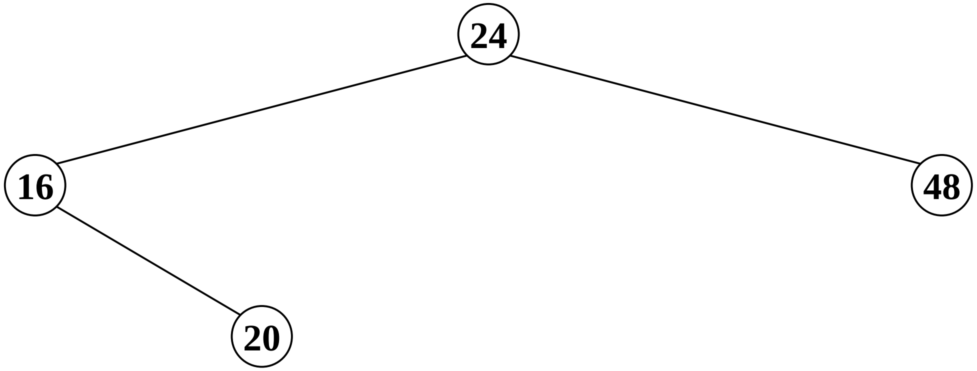
48 16 24



R/L
Dbl. Rot.

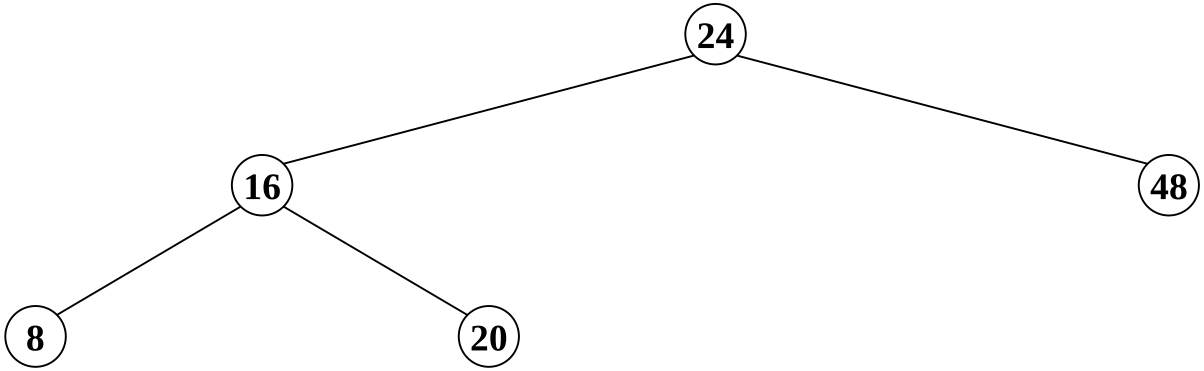
Constructing an AVL Tree – Animation

48 16 24 20



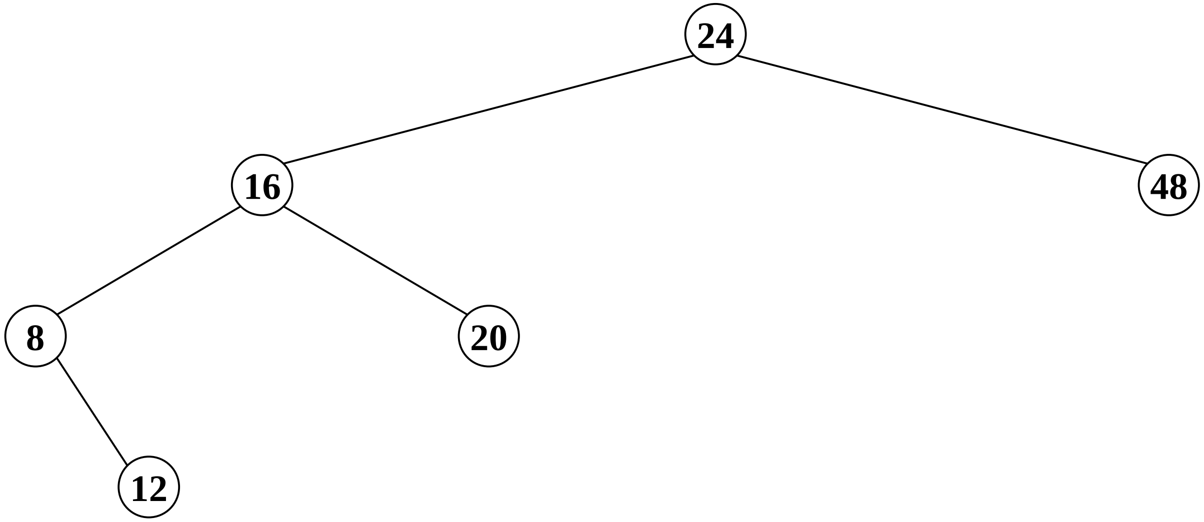
Constructing an AVL Tree – Animation

48 16 24 20 8



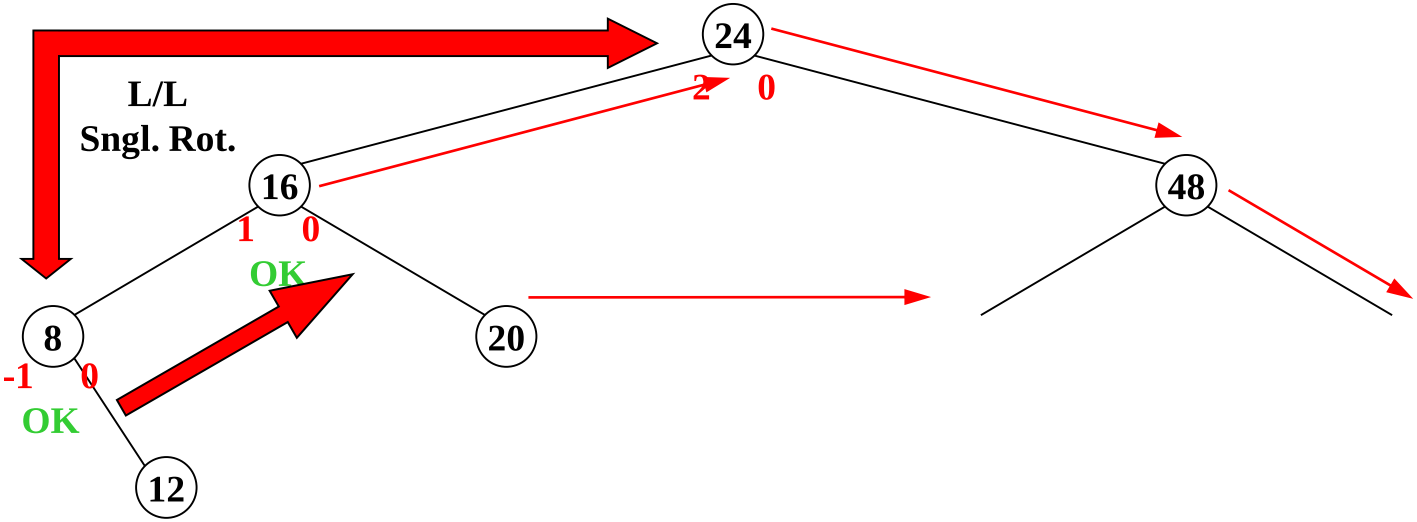
Constructing an AVL Tree – Animation

48 16 24 20 8 12



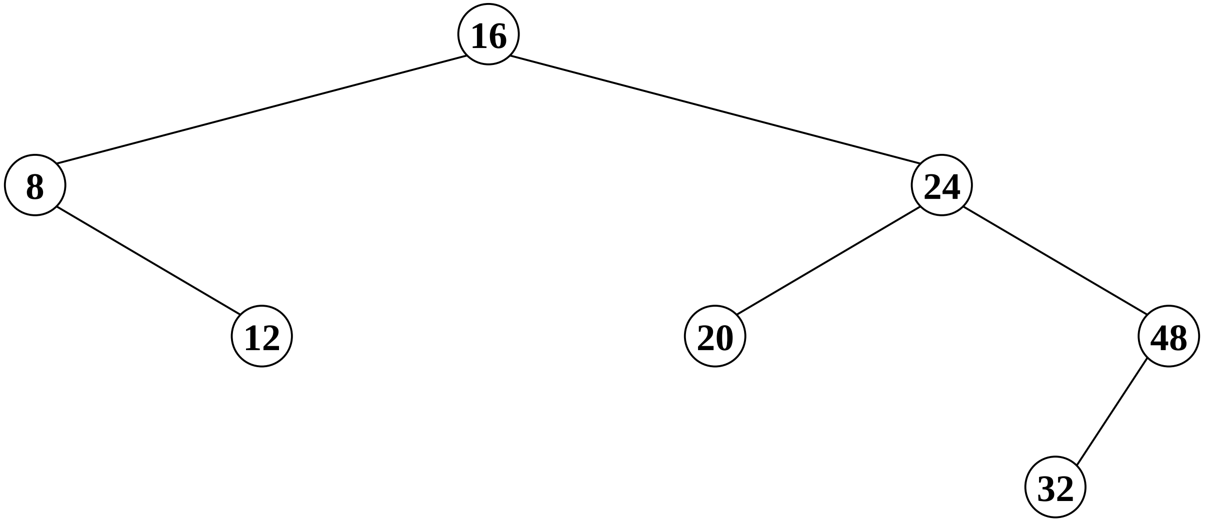
Constructing an AVL Tree – Animation

48 16 24 20 8 12



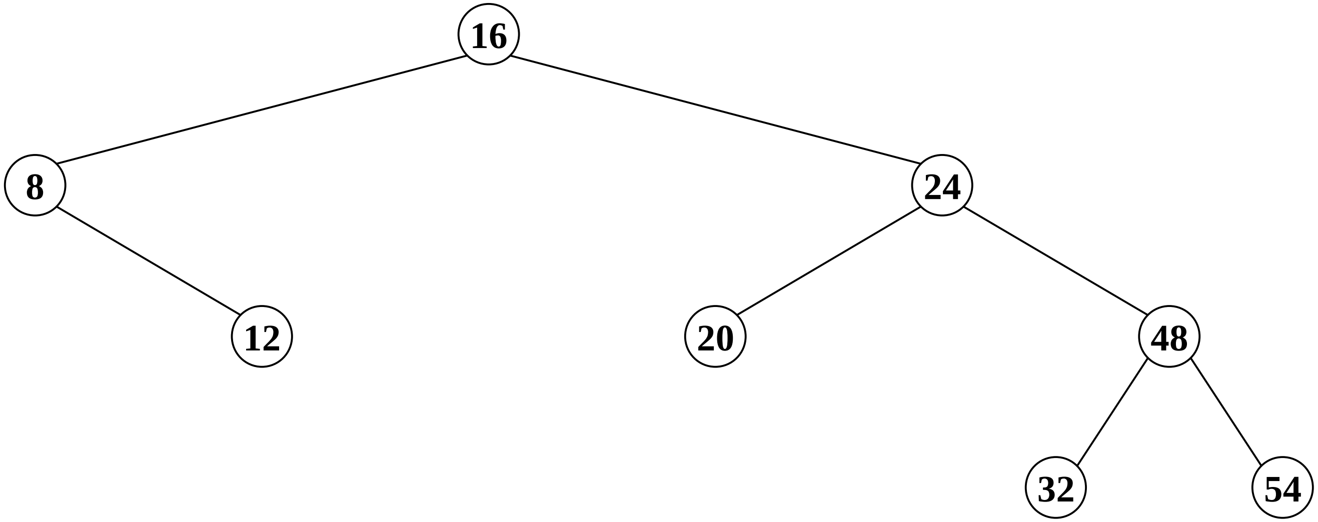
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32



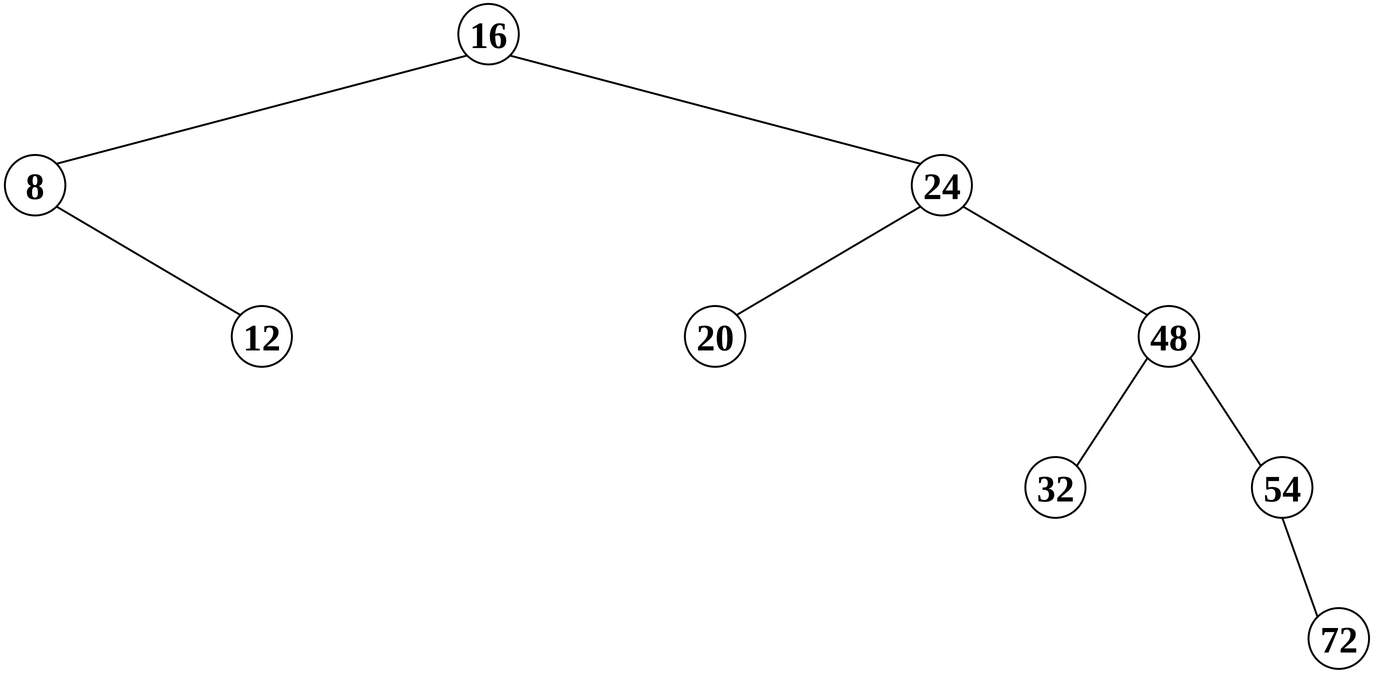
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54



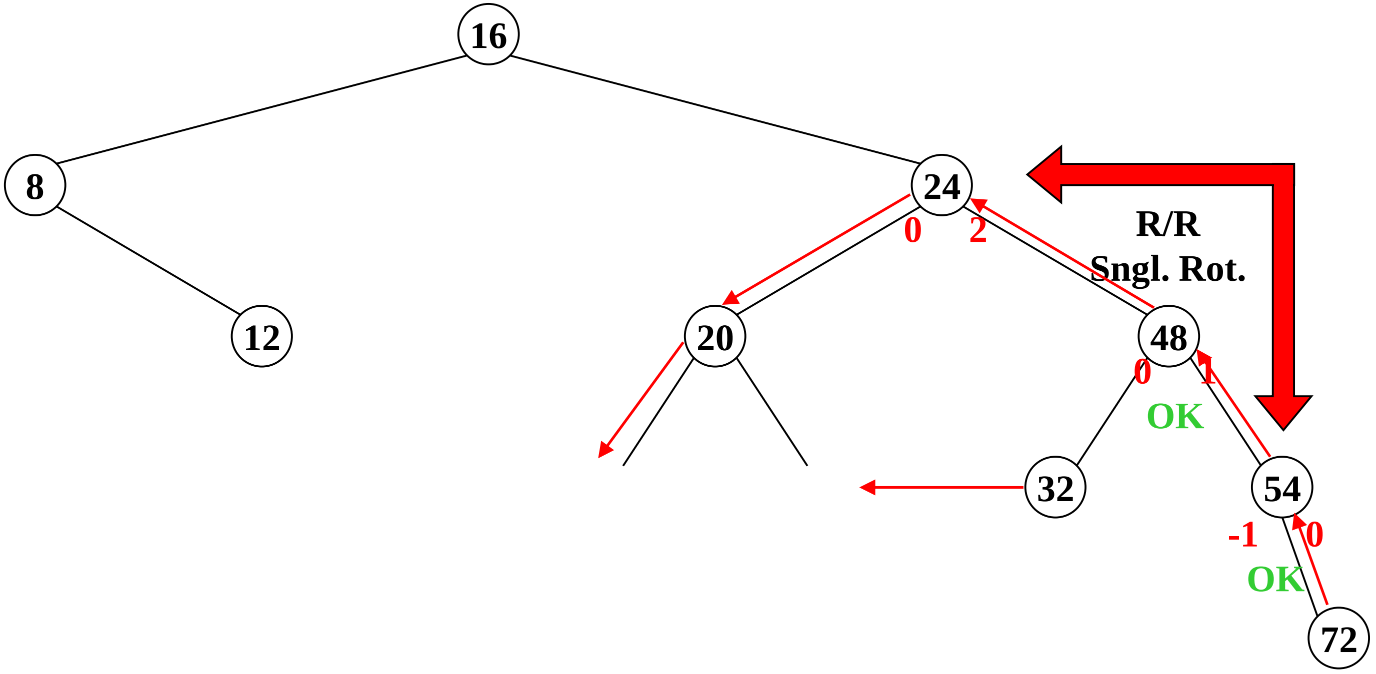
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72



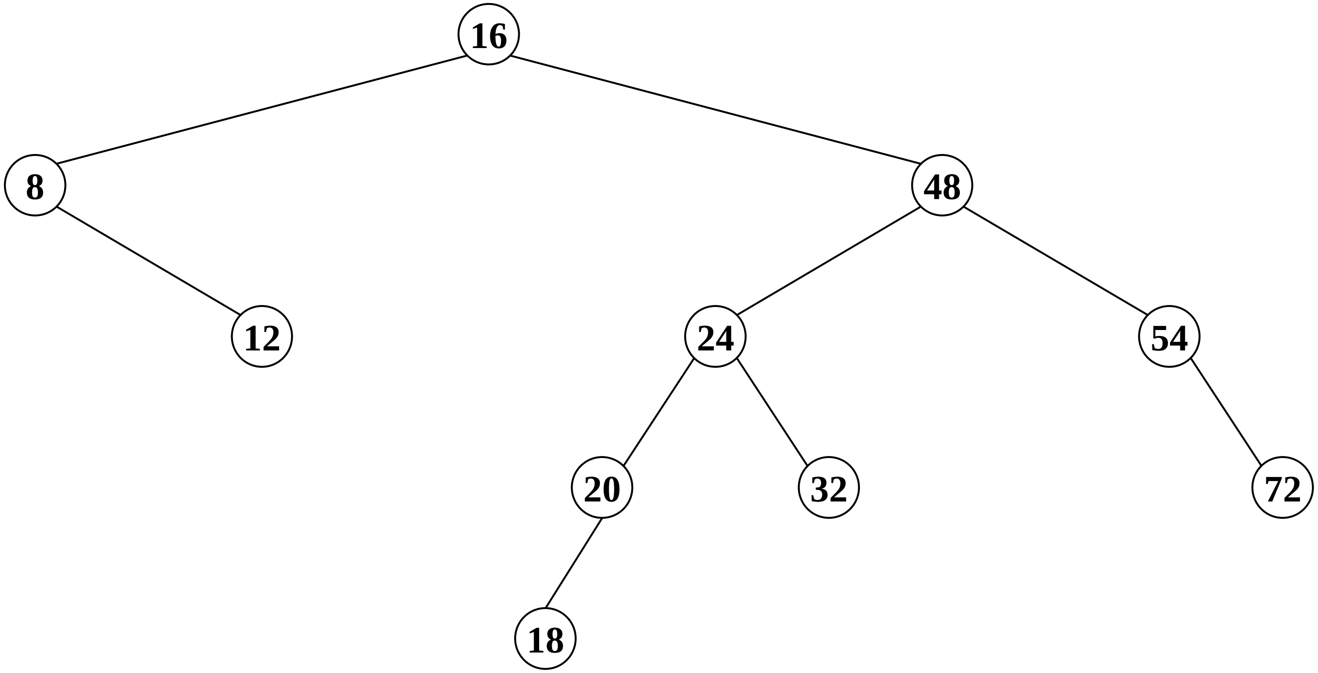
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72



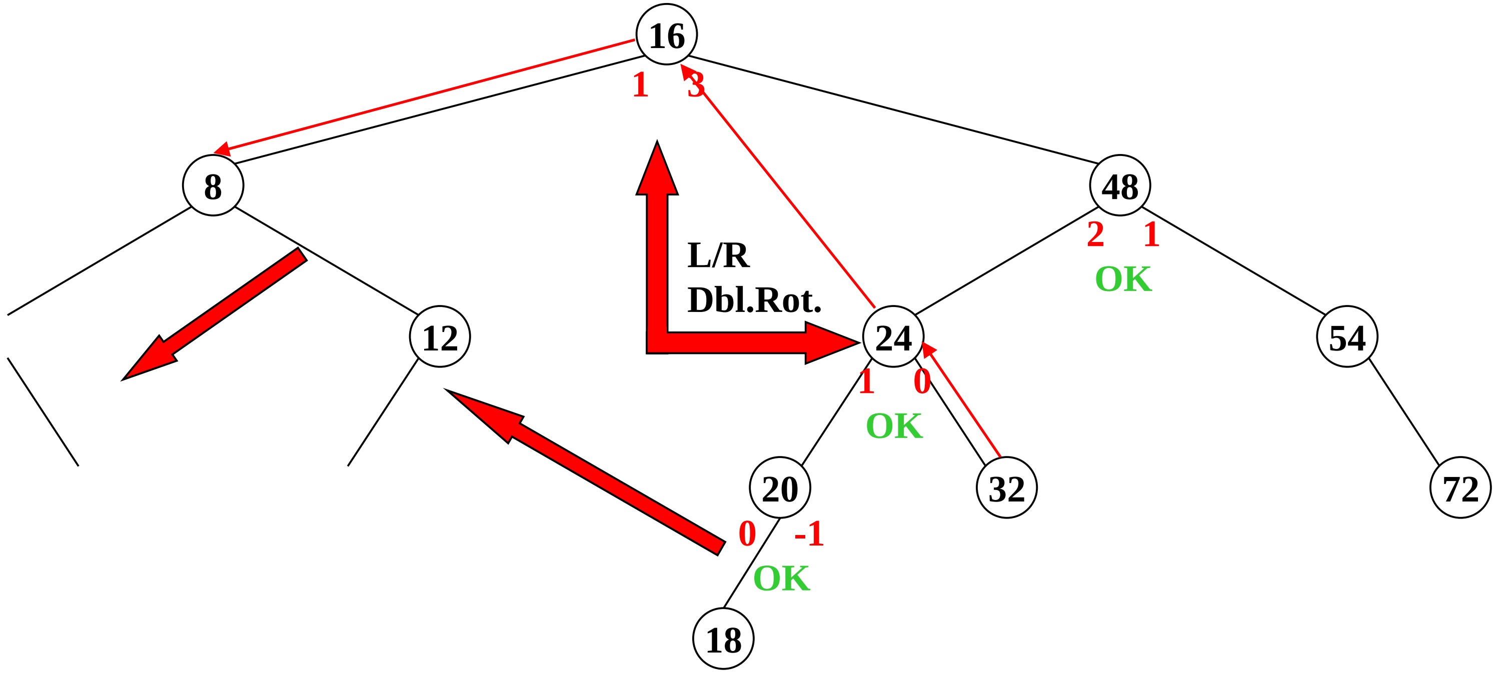
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18



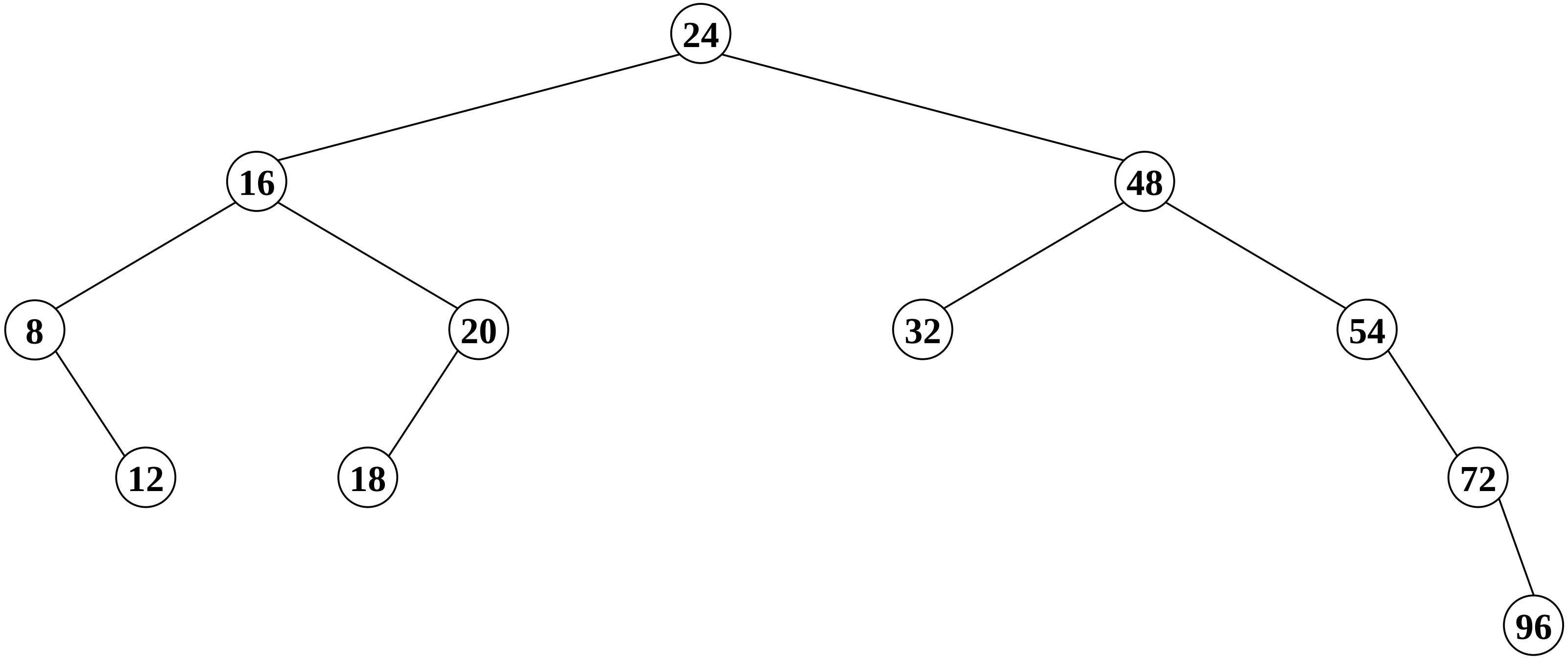
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18



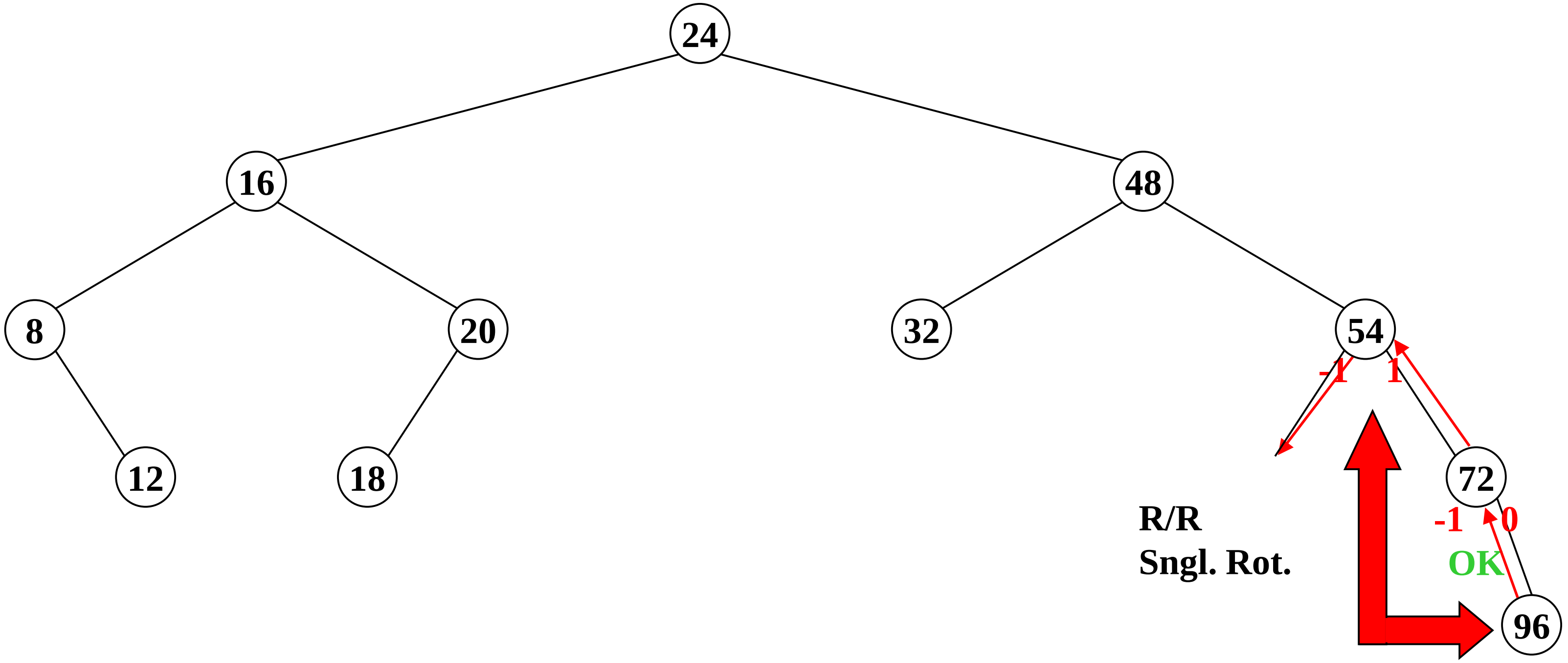
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18 96



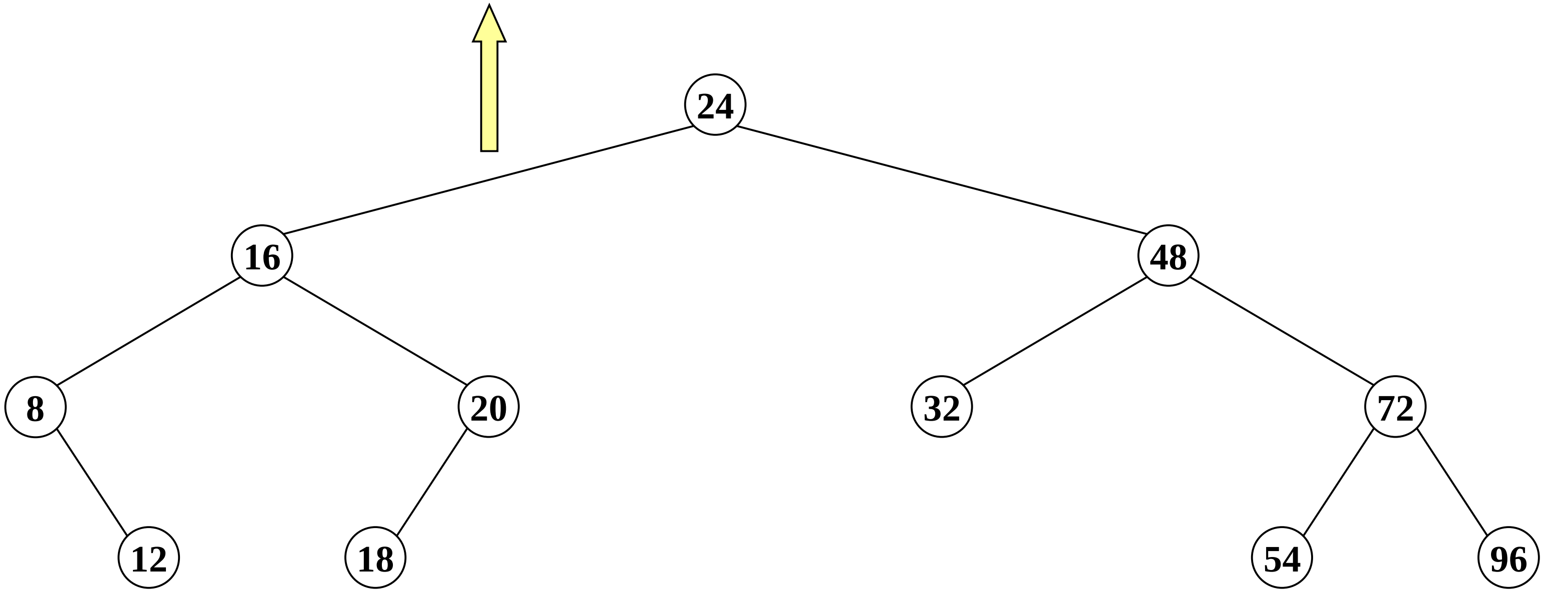
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18 96



Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18 96 64 17 60 98 68 84 36 30



Height versus Number of Nodes

- The *minimum number* of nodes in an AVL tree recursively relates to the height of the tree as follows:

$$S(h) = S(h-1) + S(h-2) + 1;$$

Initial Values: $S(0)=1; S(1)=2$

Homework: Solve for $S(h)$ as a function of h!

Splay Trees

Motivation for Splay Trees

- We are looking for a data structure where, *even though some worst case (O(n)) accesses may be possible*, *m consecutive tree operations starting from an empty tree (inserts, finds and/or removals) take* $O(m*\log_2 n)$.
- Here, the main idea is to assume that, *O(n) accesses* are not bad as long as they *occur relatively infrequently*.
- Hence, we are looking for *modifications of a BST per tree operation that attempts to minimize O(n) accesses*.

Splaying

- The underlying idea of splaying is to *move a deep node accessed upwards to the root*, assuming that it will be accessed in the near future again.
- While doing this, other deep nodes are also carried up to smaller depth levels, making the average depth of nodes closer to $O(\log_2 n)$.

Splaying

- Splaying is similar to bottom-up AVL rotations
- If a node *X is the child of the root* R,
  - then we *rotate only X and R, and this is the last rotation performed*.

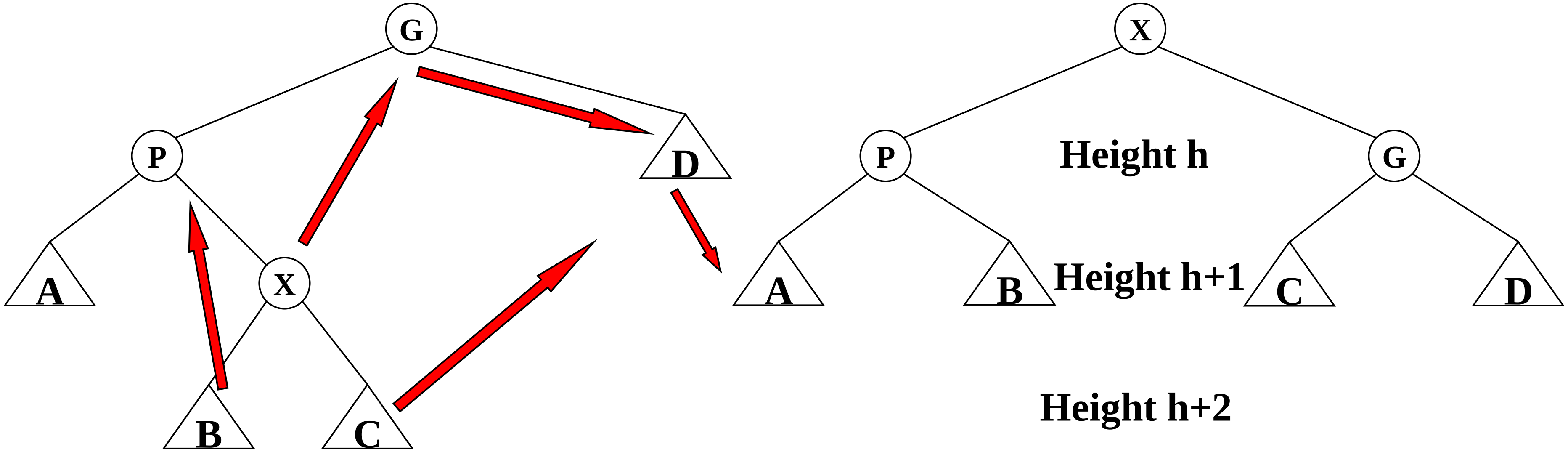
    else consider *X*, its *parent P* and *grandparent G*.

    Two cases and their symmetries to consider

    Zig-zag case, and

    Zig-zig case.

Zig-zag case



Height h

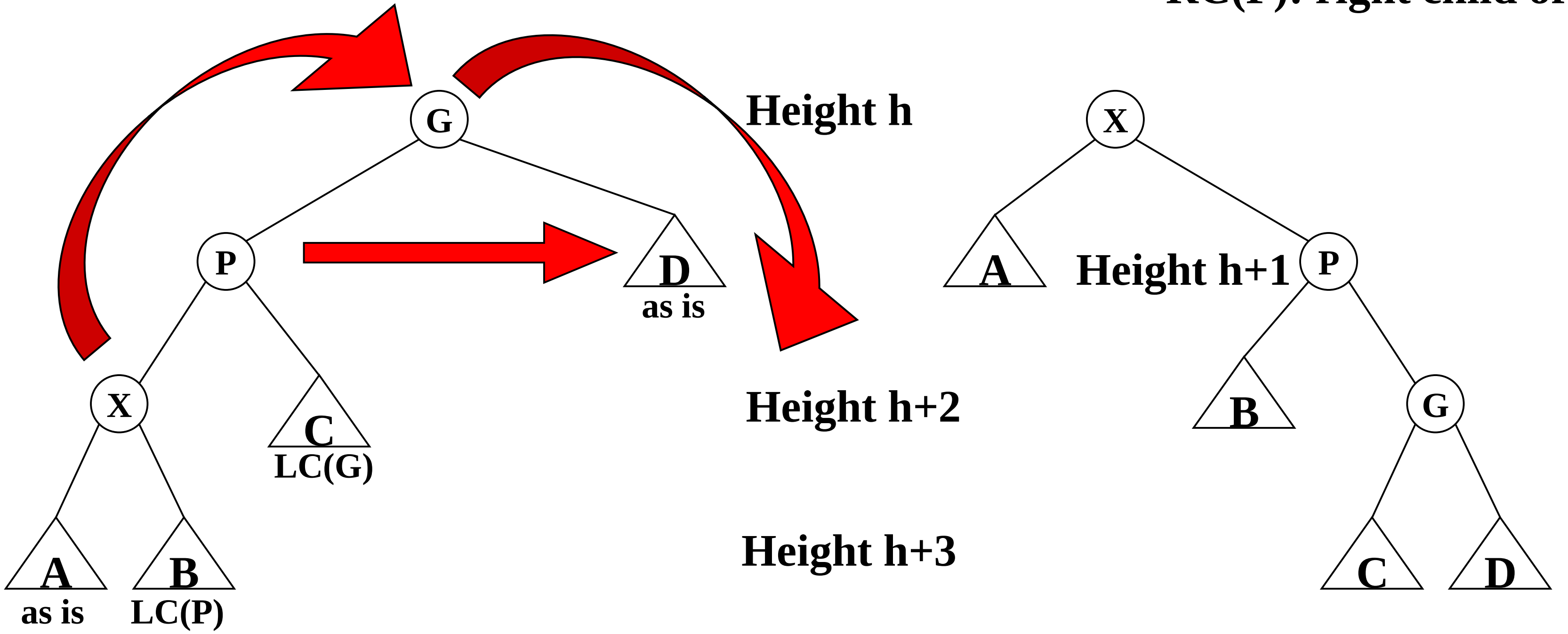
Height h+1

Height h+2

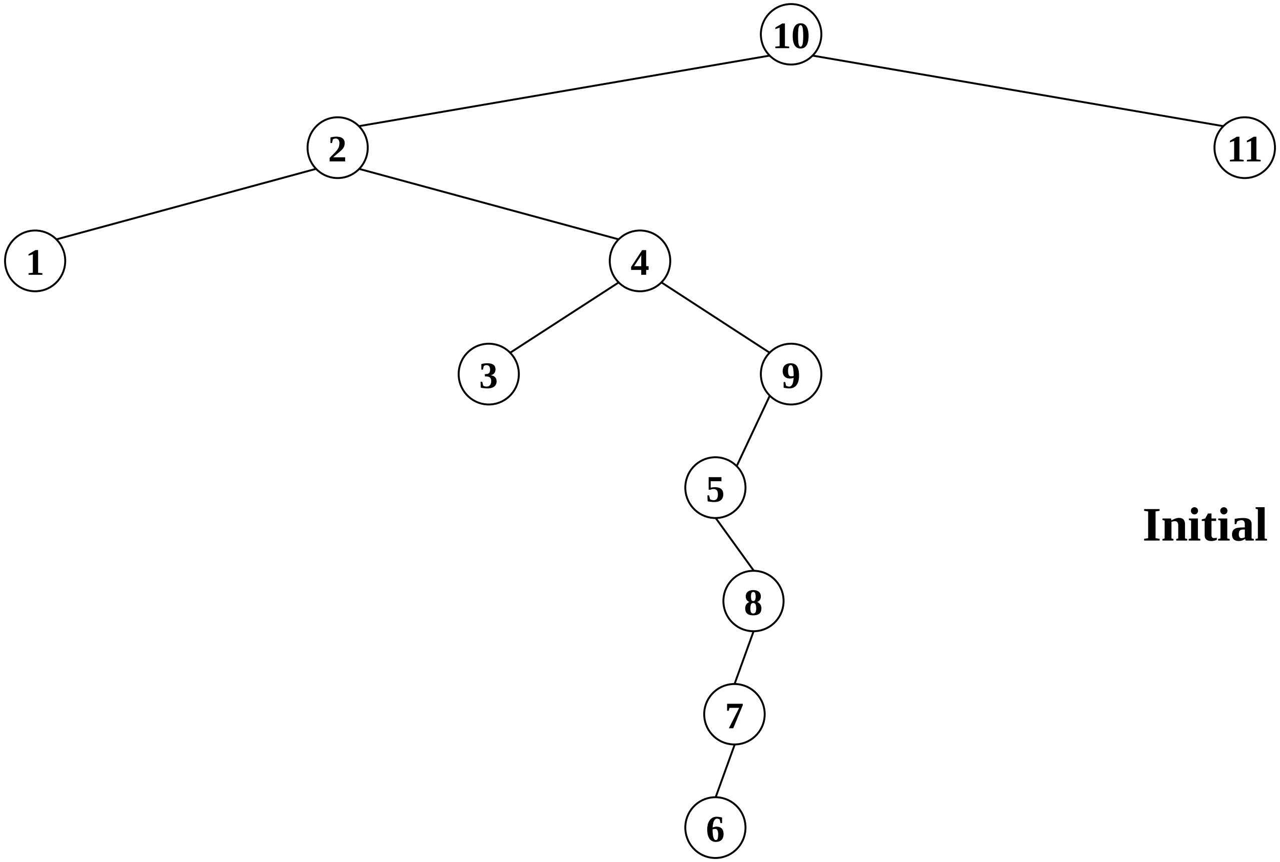
This is the same operation as an AVL double rotation in an R/L violation.

Zig-zig case

LC(P): left child of node P
RC(P): right child of node P

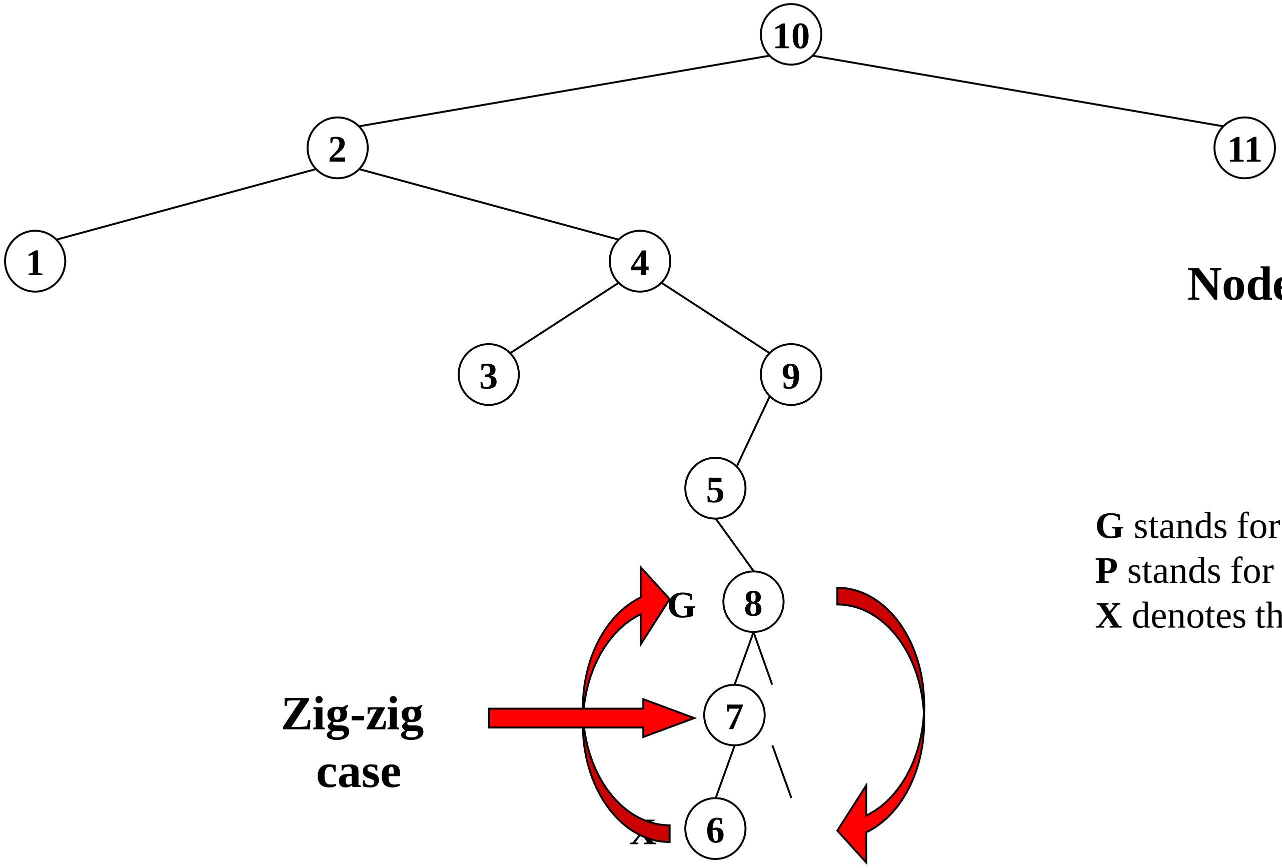


Animated Example



Initial BST

Animated Example

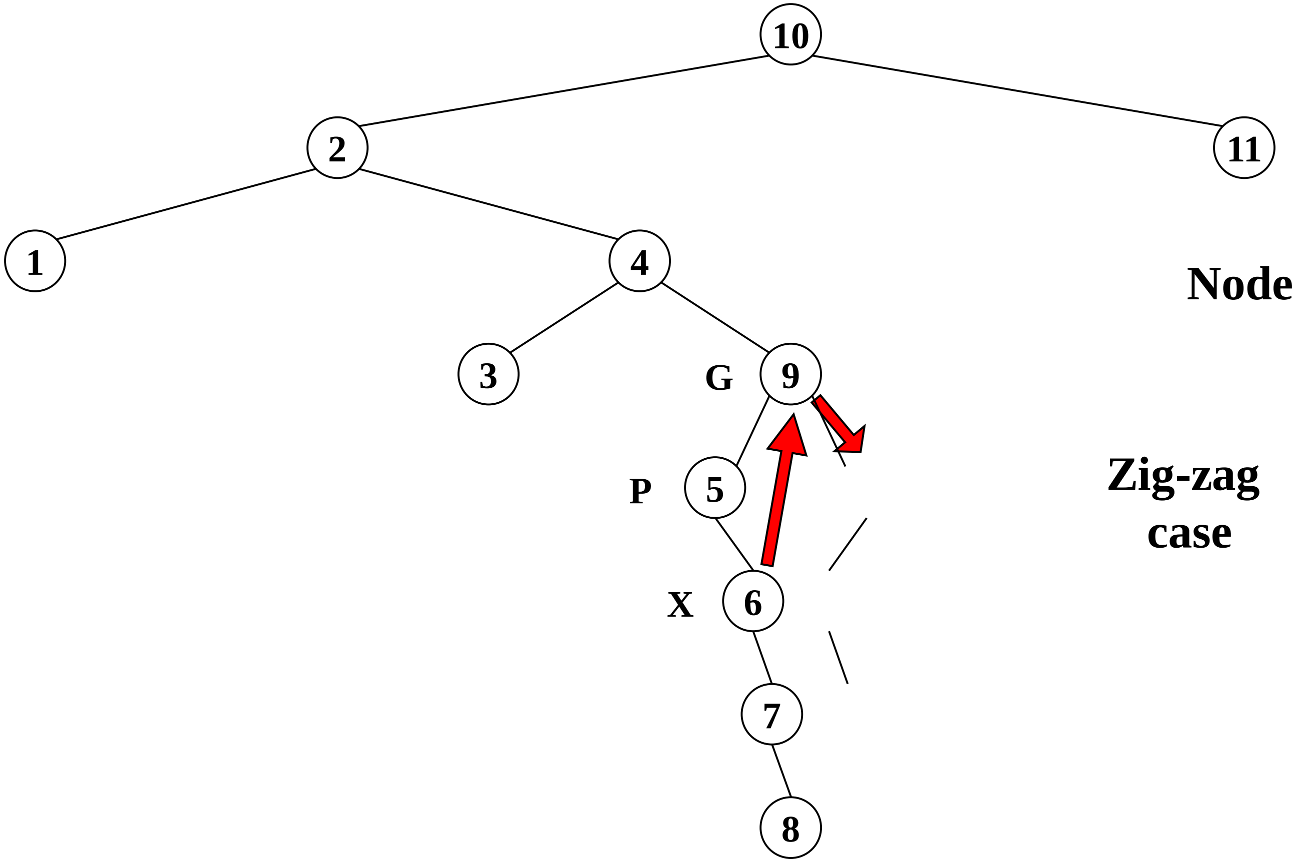


Zig-zig case

Node with 6 accessed!

G stands for **Grandparent**
P stands for **Parent**
X denotes the ***current node***

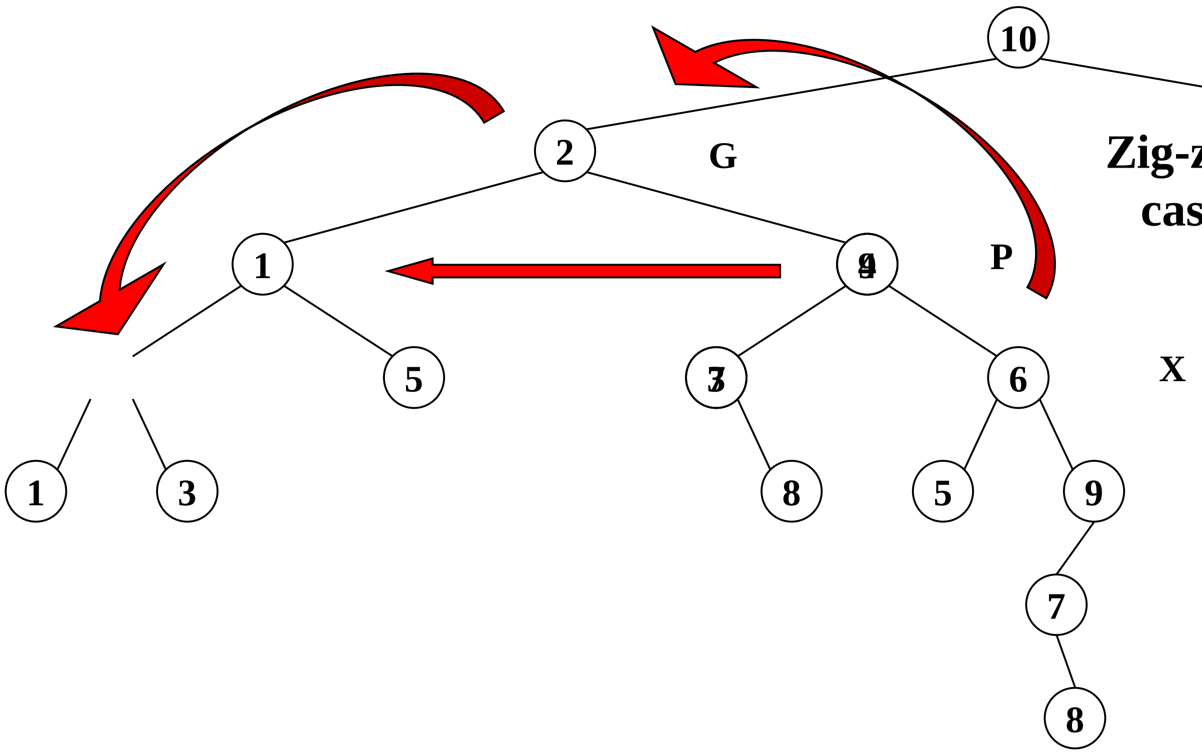
Animated Example



Node with 6 accessed!

Zig-zag case

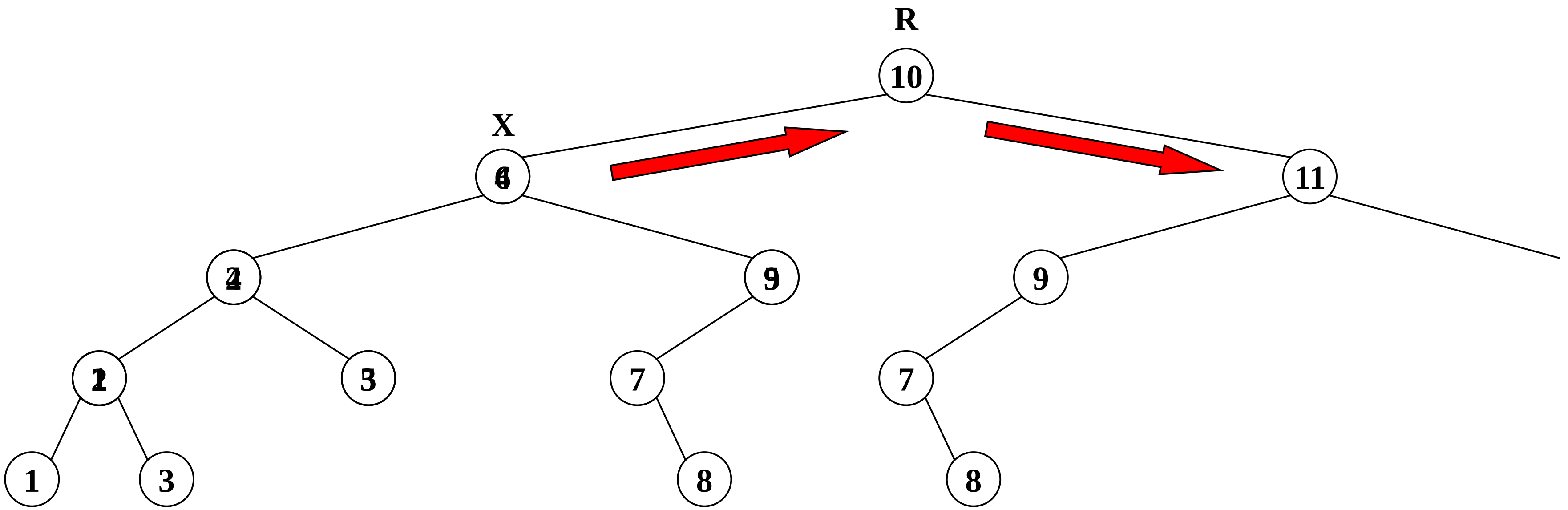
Animated Example



Zig-zig case

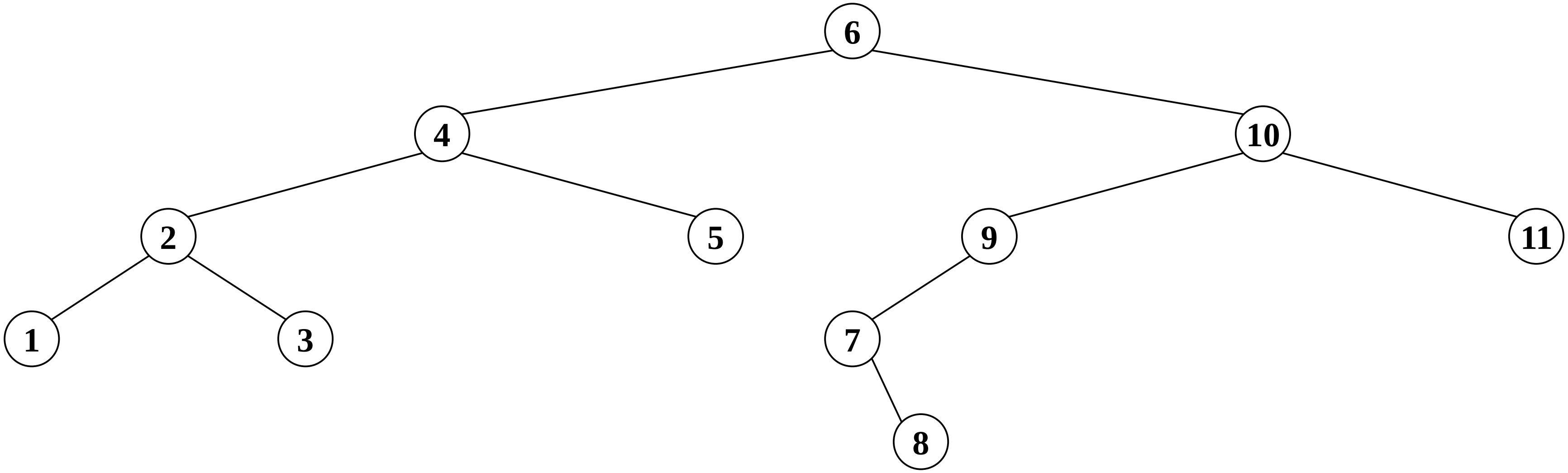
Node with 6 accessed!

Animated Example



Node with 6 accessed!

Animated Example



Node with 6 accessed!

B-Trees

Motivation for B-Trees

- Two technologies for providing memory capacity in a computer system
  - *Primary* (main) *memory (silicon chips)*
  - *Secondary storage (magnetic disks)*
- Primary memory
  - 5 orders of magnitude (i.e., about 10^5 times) *faster*,
  - 2 orders of magnitude (about 100 times) *more expensive*, and
  - by at least 2 orders of magnitude *less in size*

than secondary storage due to mechanical operations involved in magnetic disks.

Motivation for B-Trees

- During one disk read or disk write ((4-8.5msec for 7200 RPM sequential disks (not SSDs!)), MM can be accessed about 10^5 times (100 nanosec per access).
- To reimburse (compensate) for this time, at each disks access, *not a single item*, but one or more ***equal-sized** pages* of items (each page 2^{11}-2^{14} bytes) are accessed.
- We need some data structure to store these ***equal sized** pages* in MM.
- *B-Trees*, with their ***equal-sized leaves** (as big as a page),* are suitable data structures for storing and performing regular operations on paged data.

B-Trees

- A *B-tree* is a rooted tree with the following properties:
- Every node *x* has the following fields:
  - *n[x]*, the number of keys currently stored in *x*.
  - the *n[x]* keys themselves, in *non-decreasing order*, so that

$$key_1[x] \le key_2[x] \le \ldots \le key_{n[x]}[x]\ ,$$

  - *leaf[x]*, a boolean value, true if *x* is a leaf.

B-Trees

- *Each internal (non-leaf) node* has $n[x]+1$ pointers, $c_1[x],\ldots, c_{n[x]+1}[x]$, to its children. *Leaf nodes* have *no children*, hence no pointers!
- The keys separate the ranges of keys stored in each subtree: if k_i is any key stored in the subtree with root $c_i[x]$, then

$$k_1 \leq key_1[x] \leq k_2 \leq key_2[x] \leq \ldots \leq key_{n[x]}[x] \leq k_{n[x]+1} \,.$$

- *All leaves have the same depth*, *h*, equal to the *tree's height*.

B-Trees

- There are lower and upper bounds on the number of keys a node may contain. These bounds can be expressed in terms of a fixed integer $t \geq 2$ called the *minimum degree* of the B-Tree.
  - Lower limits
    - All *nodes but the root* has *at least t-1* keys.
    - Every *internal node but the root* has *at least **t** children*.
    - A non-empty tree's **root** must have *at least one key*.

B-Trees

- Upper limits
  - Every *node* can contain *at most* ***2t-1 keys***.
  - Every *internal node* can have *at most* ***2t children***.
  - A node is defined to be full if it has exactly *2t-1 keys*.
- For a *B-tree* of minimum degree $t \geq 2$ and n nodes

$$h \leq \log_t \frac{n+1}{2}$$

Basic Operations on B-Trees

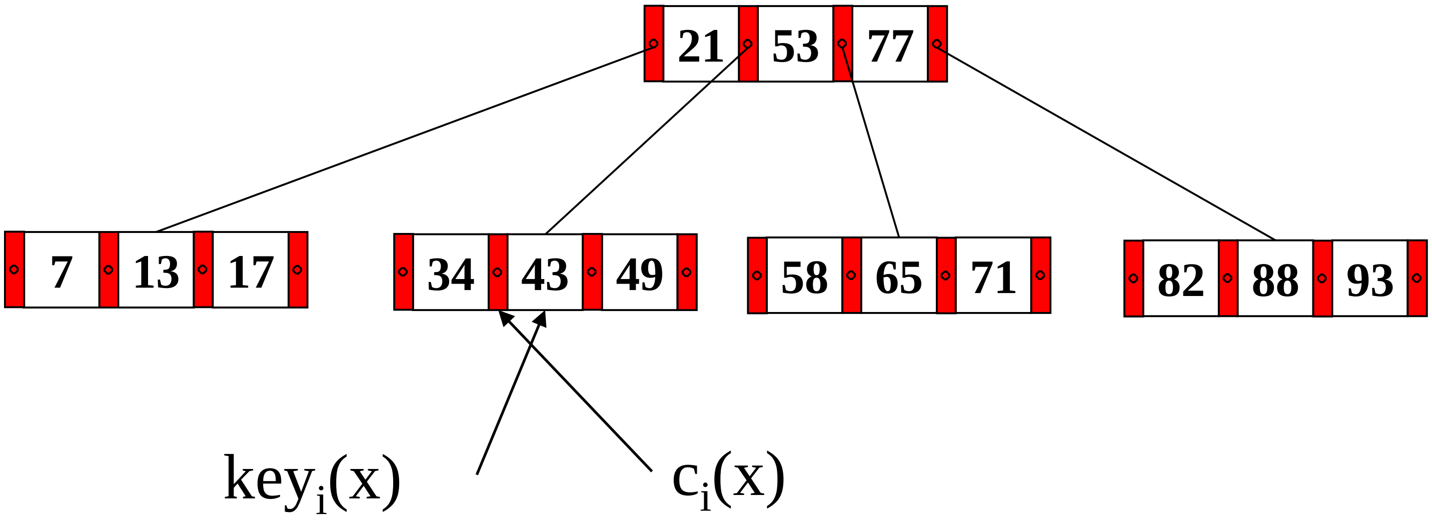
- *B-tree search*
- *B-tree insert*
- *B-tree removal*

Disk Operations in B-Tree operations

- Suppose *x* is a pointer to an object.
- It is accessible if it is in the main memory.
- If it is on the disk, it needs to be transferred to the main memory to be accessible. This is done by *DISK_READ(x)*.
- To save any changes made to any field(s) of the object pointed to by *x*, a *DISK_WRITE(x)* operation is performed.

Search in B-Trees

- Similar to search in BSTs with the exception that instead of a binary, a multi-way ($n[x]$+1-way) decision is made.



Search in B-Trees

```
B-tree-Search(x,k)
{  i=1;
   while (i ≤ n[x] and k > key_i[x]) i++;
   if (i ≤ n[x] and k = key_i[x])               // if key found
         return (x,i);
   if (leaf[x])                        // if key not found at a leaf
         return NULL;
   else {DISK_READ(c_i[x]);                     // if key < key_i[x]
          return B-tree-Search(c_i[x],k);}
}
```

Insertion in B-Trees

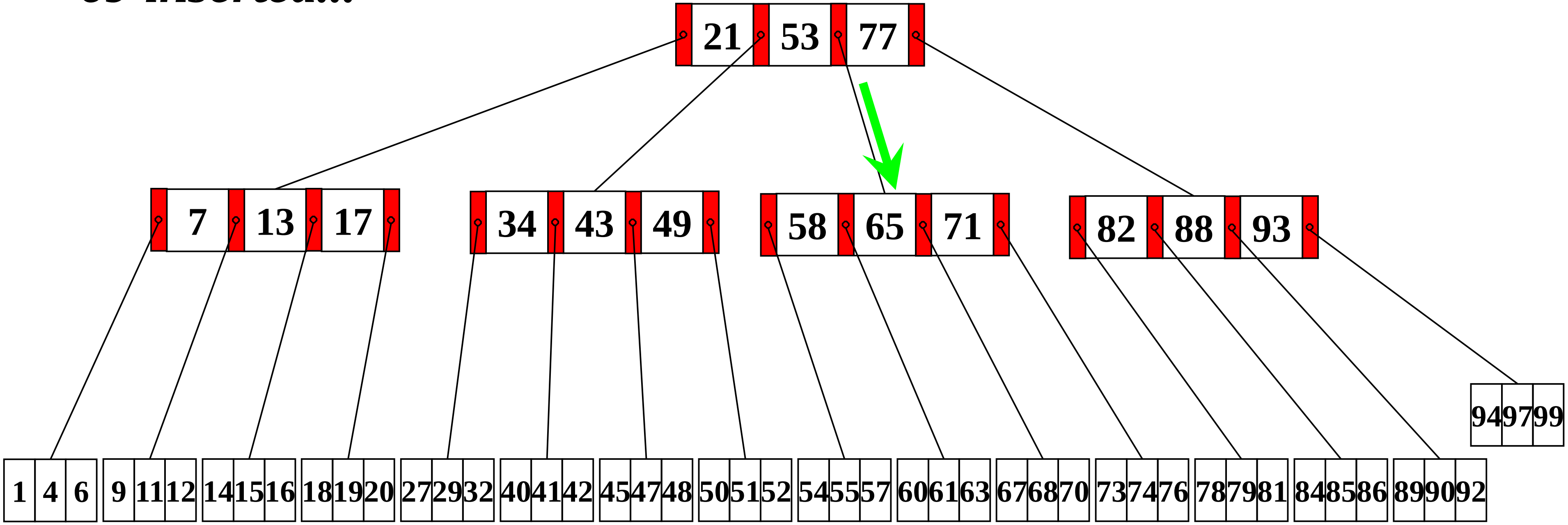
- Insertion into a B-tree is more complicated than that into a BST, since the creation of a new node to place the new key may violate the B-tree property of the tree.
- Instead, the key is put *into a leaf node x if it is not full*.
- If full, a *split* is performed, which splits a full node (with *2t-1* keys) at its *median key*, $key_t[x]$, into two nodes with *t-1* keys each.
- $key_t[x]$ moves up into the parent of *x* and identifies the split point of the two new trees.

Insertion in B-Trees

- A *single-pass insertion* starts at the root traversing *down to the leaf* into which the key is to be inserted.
- On the path down, *all full nodes are split* including a full leaf that also guarantees a parent with an available position for the median key of a full node to be placed.

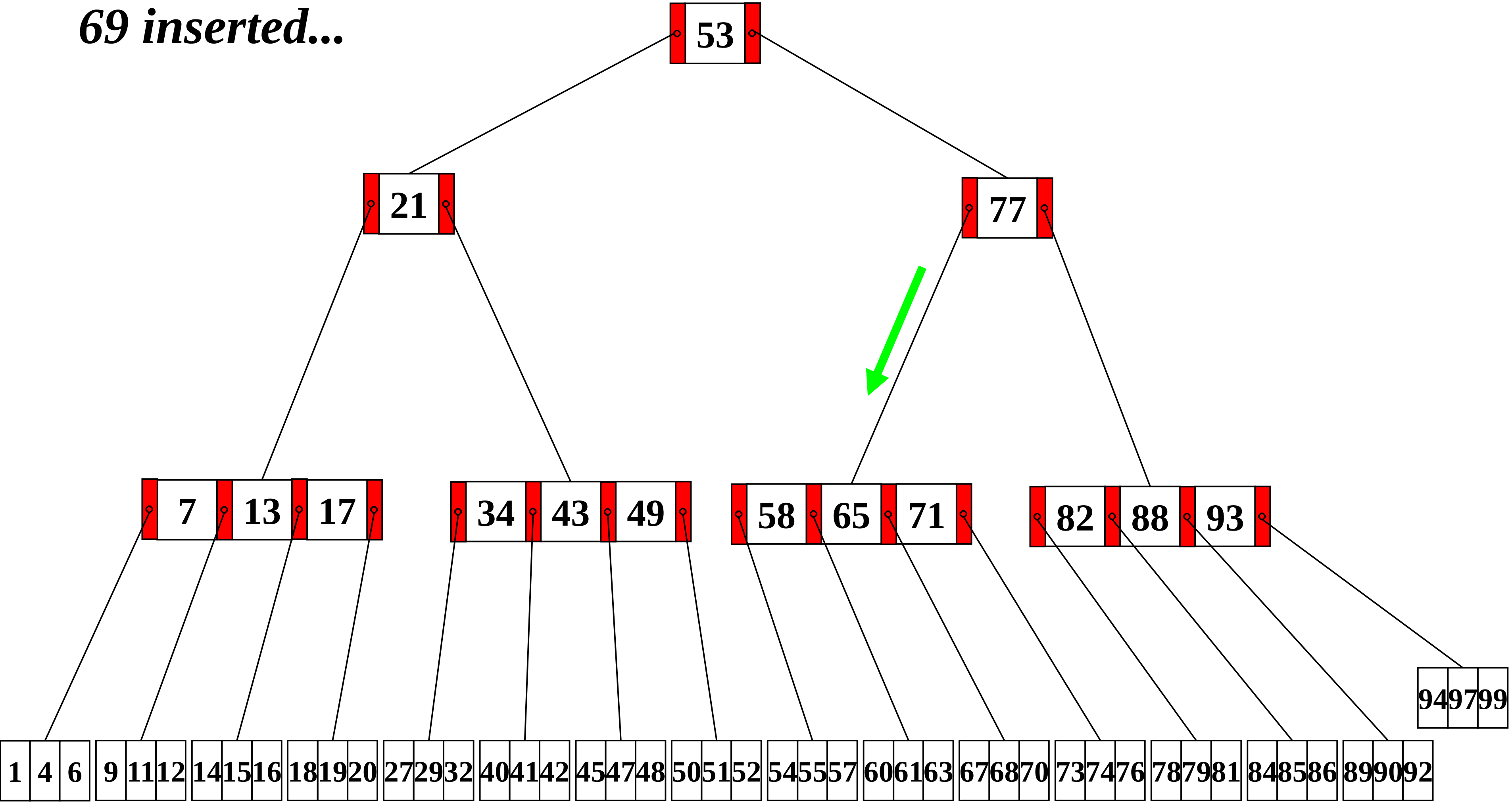
Insertion in B-Trees: Example

69 inserted...



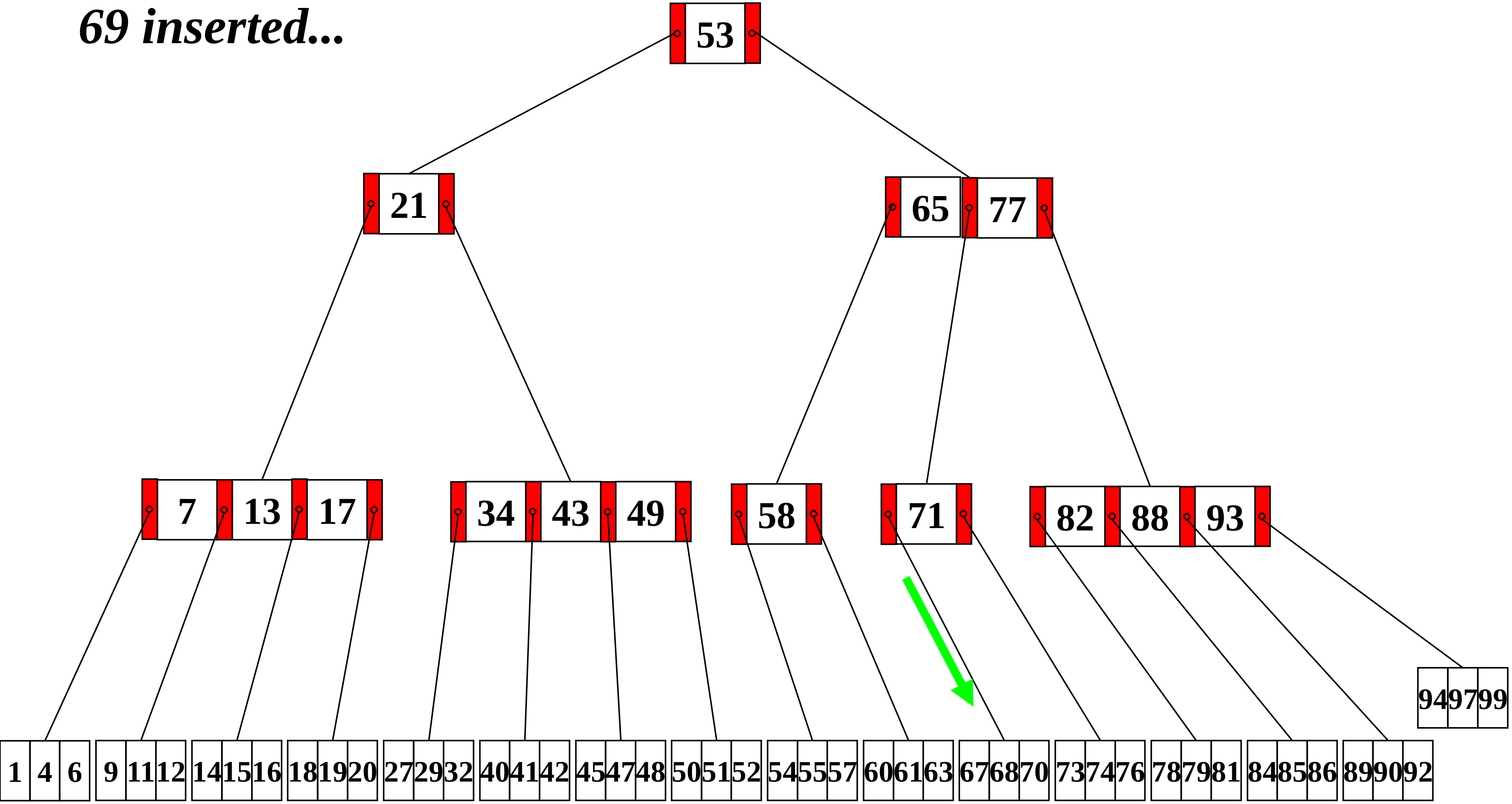
Insertion in B-Trees: Example

69 inserted...



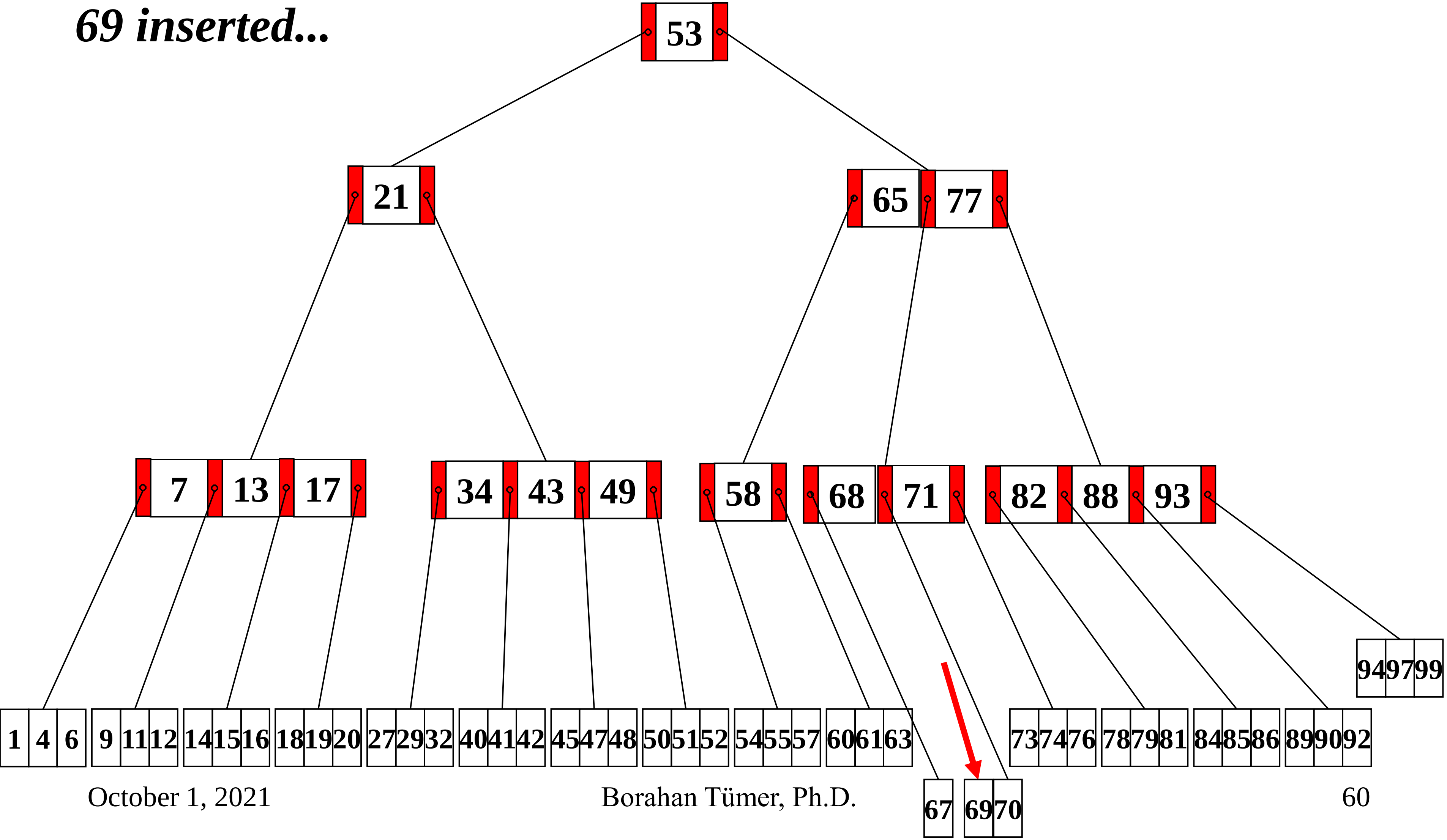
Insertion in B-Trees: Example

69 inserted...



Insertion in B-Trees: Example

69 inserted...



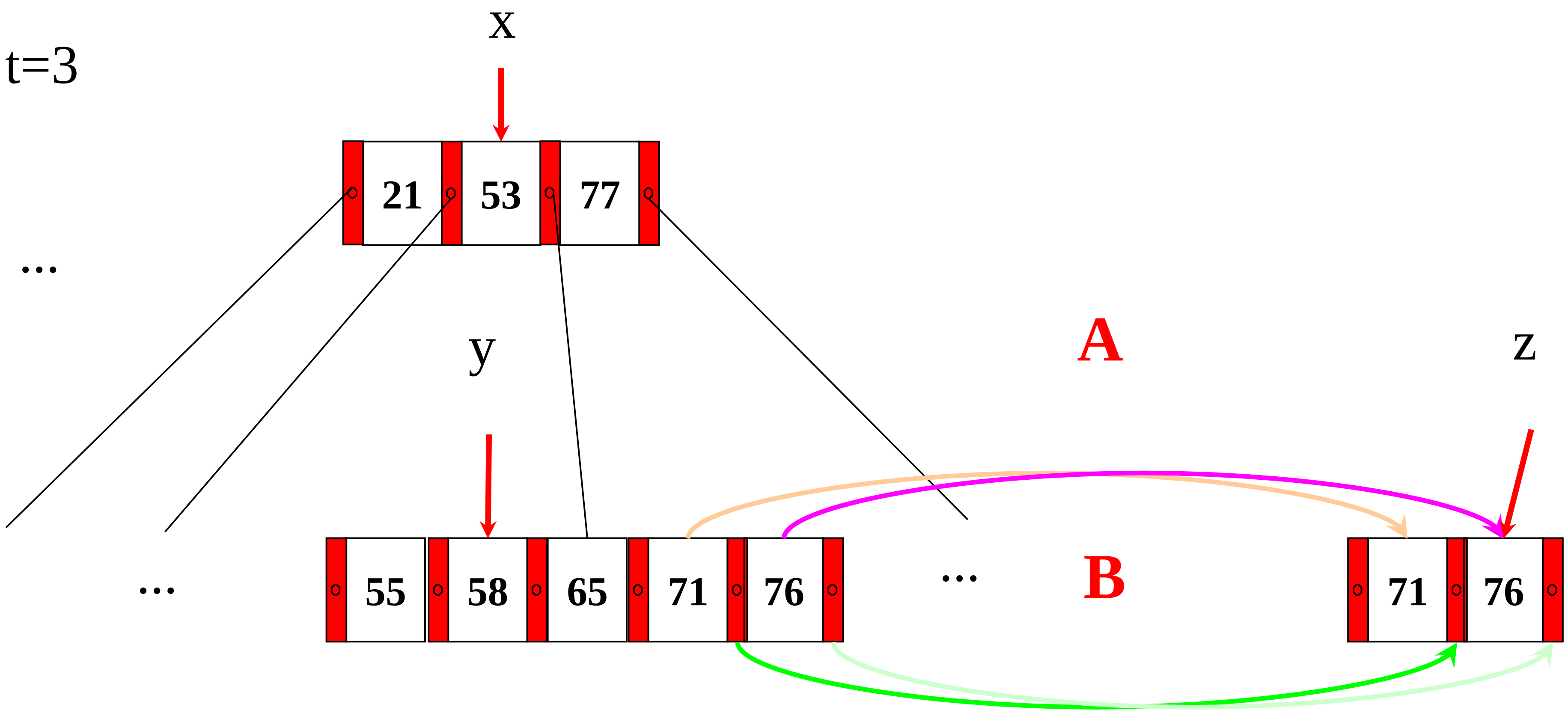
Insertion in B-Trees:B-tree-Insert

```
B-tree-Insert(T,k)
{  r=root[T];
   if (n[r] == 2t-1) {
         s=malloc(new-B-tree-node);
         root[T]=s;
         leaf[s]=false;
         n[s]=0;
         $c_1$[s]=r;
         B-tree-Split-Child(s,1,r);
         B-tree-Insert-Nonfull(s,k); }
   else  B-tree-Insert-Nonfull(r,k);
}
```

Insertion in B-Trees:B-tree-Split-Child

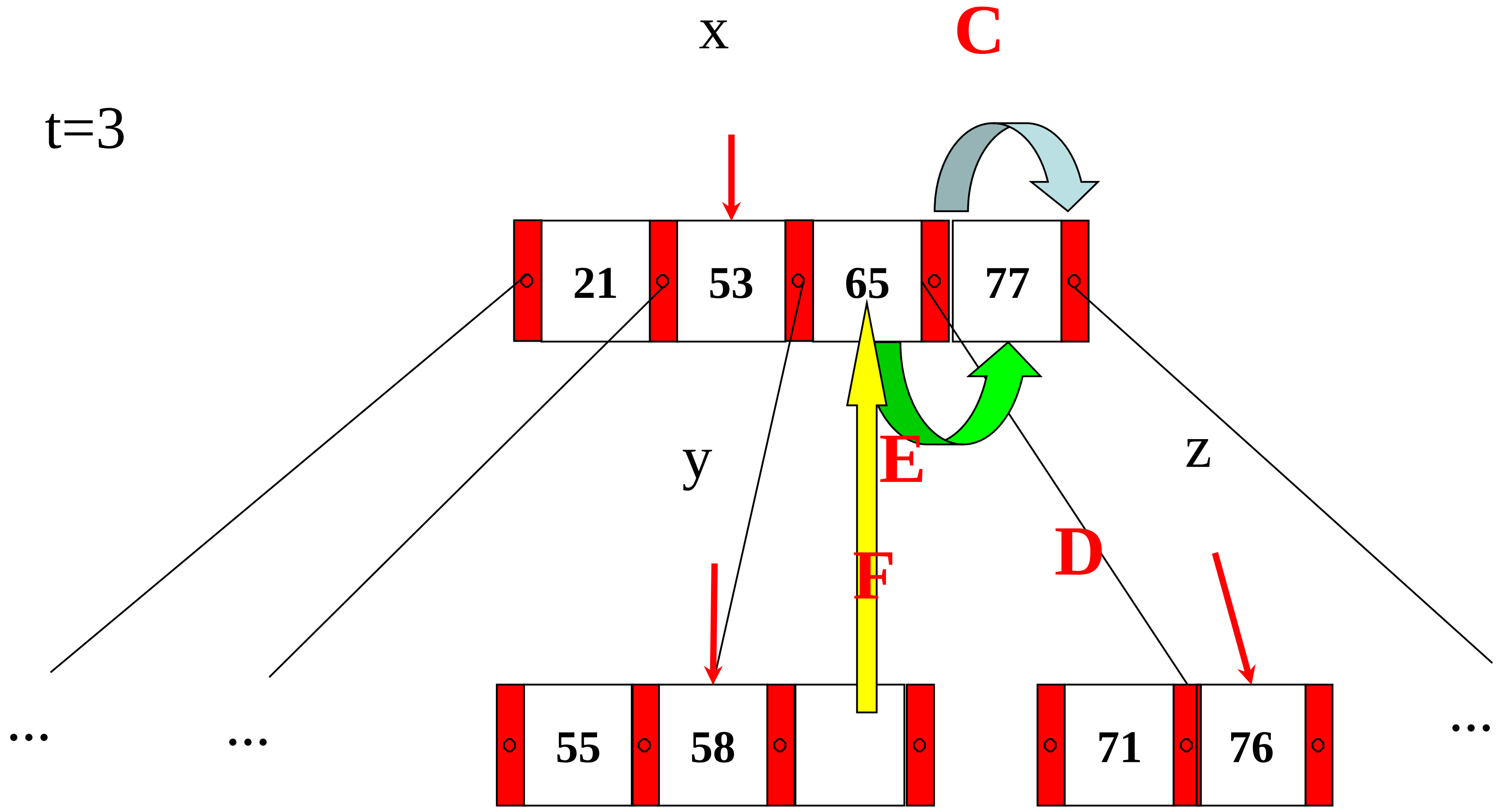
```
B-tree-Split-Child(x,i,y)
{   z=malloc(new-B-tree-node);
    leaf[z]=leaf[y];
    n[z]=t-1;
    for (j = 1; j < t) $key_j$[z]=$key_{j+t}$[y];            A
    if (!leaf[y])
        for (j = 1; j <= t;j++) $c_j$[z]=$c_{j+t}$[y];       B
    n[y]=t-1;
    for (j=n[x]+1; j>=i+1; j--) $c_{j+1}$[x]=$c_j$[x];       C
    $c_{i+1}$[x]=z;                                          D
    for (j=n[x]; j>=i; j--) $key_{j+1}$[x]=$key_j$[x];       E
    $key_i$[x]=$key_t$[y]; n[x]++;                           F
    DISK_WRITE(y);
    DISK_WRITE(z);
    DISK_WRITE(x);
}
```

B-tree-Split-Child: Example

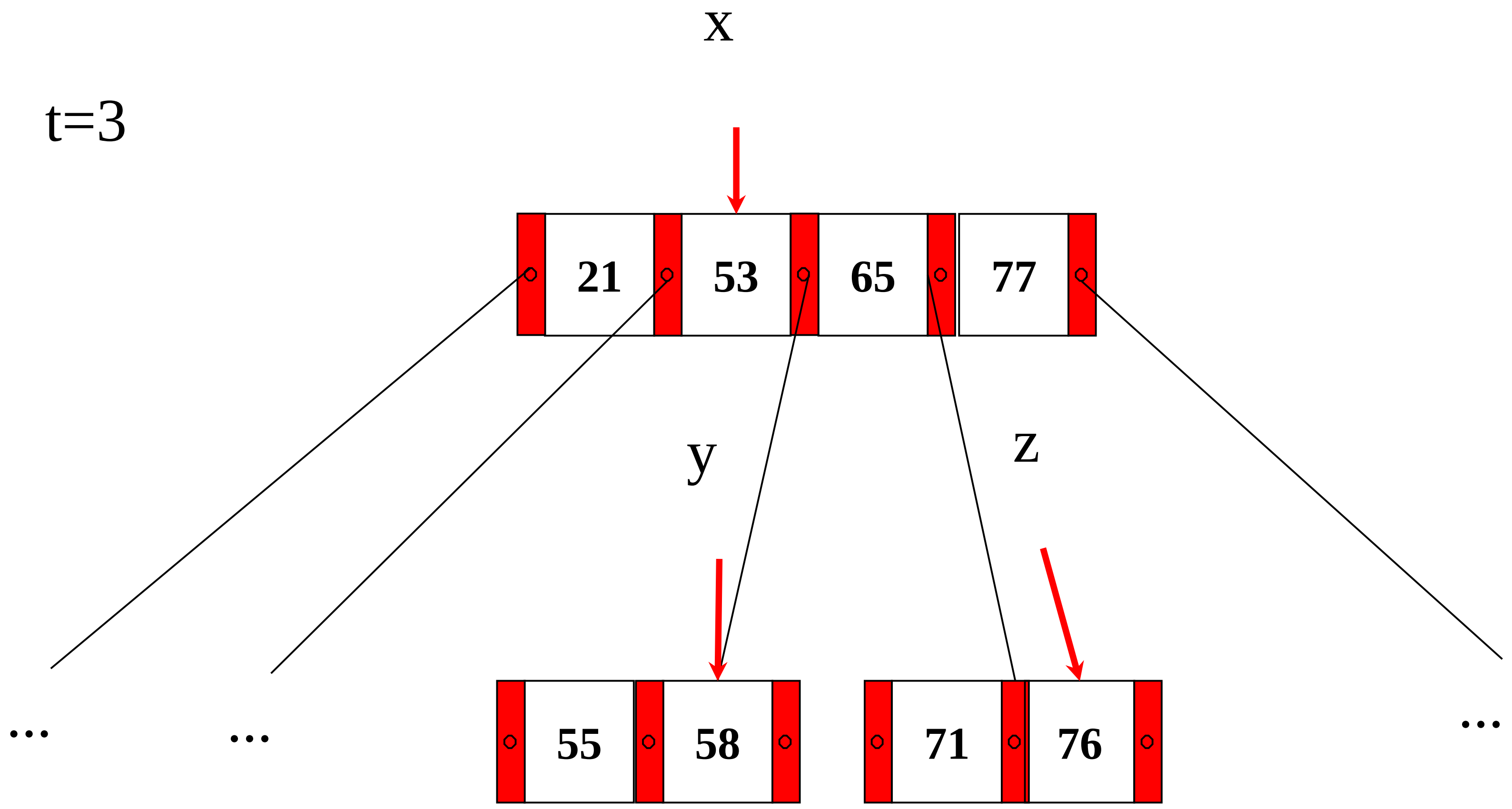


B-tree-Split-Child: Example

t=3



B-tree-Split-Child: Example



Insertion in B-Trees: B-tree-Insert-Nonfull

```
B-tree-Insert-Nonfull(x,k)
{   i=n[x];
    if (leaf[x]) {
            while (i≥1 and k < $key_i[x]$) {$key_{i+1}[x]=key_i[x]$; i--;}
            $key_{i+1}[x]$=k;
            n[x]++;
            DISK_WRITE(x);
     }
    else {
             while (i≥1 and k < $key_i[x]$) i--;
             i++;
             DISK_READ($c_i[x]$);
             if (n[$c_i[x]$]==2t-1) {
                         B-tree-Split-Child(x,i, $c_i[x]$);
                         if (k > $key_i[x]$) i++;
             }
             B-tree-Insert-Nonfull($c_i[x]$,k);
          }
}
```

```
if x is a leaf
  then place key in x;
       write x on disk;
  else find the node (root of
       subtree) key goes to;
       read node from disk;
       if node full
         split node at key's
         position;
      recursive call with
      node split and key;
```

Removing a key from a B-Tree

- Removal in B-trees is different than insertion only in that *a key may be removed from any node, not just from a leaf*.
- As the insertion algorithm splits any full node down the path to the leaf to which the key is to be inserted, a recursive removal algorithm may be written to ensure that for any call to removal on a node x, the number of keys in x is at least the minimum degree t.

Various Cases of Removing a key from a B-Tree

1. If the key *k* is in node *x* and *x* is a leaf, remove the key *k* from *x*.
2. If the key *k* is in node *x* and *x* is an internal node, then
    a. If the child *y* that precedes *k* in node *x* has at least *t* keys, then find the predecessor *k'* of *k* in the subtree rooted at *y*. Recursively delete *k'*, and replace *k* by *k'* in *x*. Finding *k'* and deleting it can be performed in a single downward pass.

Various Cases of Removal a key from a B-Tree

b. Symmetrically, if the child *z* that follows *k* in node *x* has at least *t* keys, then find the successor *k'* of *k* in the subtree rooted at *z*. Recursively delete *k'*, and replace *k* by *k'* in *x*. Finding *k'* and deleting it can be performed in a single downward pass.

c. Otherwise, if both *y* and *z* have only *t-1* keys, merge *k* and all of *z* into *y* so that x loses both *k* and the pointer to *z* and *y* now contains *2t-1* keys. Free *z* and recursively delete *k* from *y*.

Various Cases of Removal a key from a B-Tree

3. If *k* is not present in internal node *x*, determine root $c_i[x]$ of the subtree that must contain *k*, if *k* exists in the tree. If $c_i[x]$ has only t-*1* keys, execute step *3a* or *3b* as necessary to guarantee that we descend to a node containing at least *t* keys. Then finish by recursing on the appropriate child of *x*.

Various Cases of Removal a key from a B-Tree

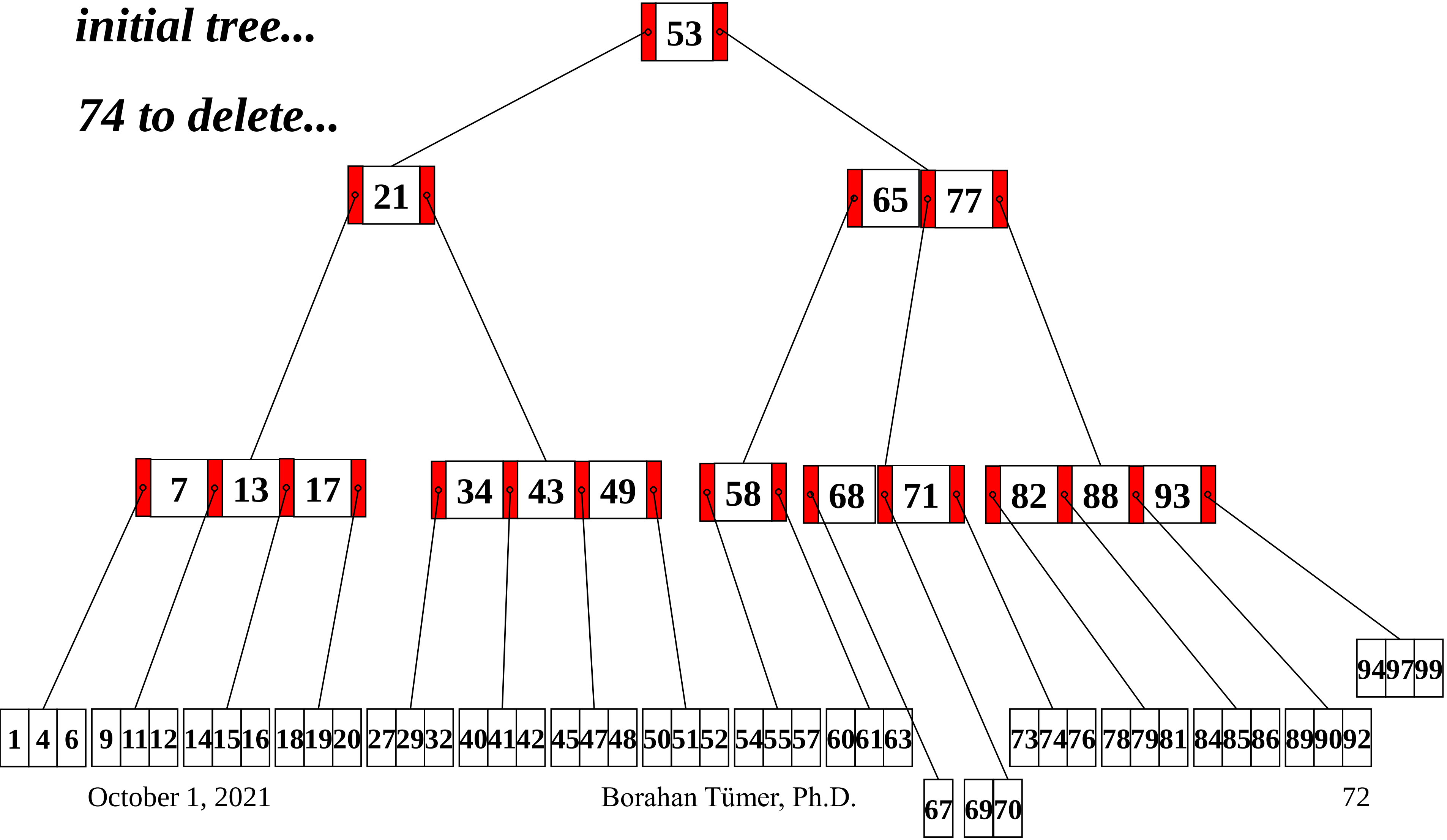
a. If $c_i[x]$ has only t-1 keys but has an immediate sibling with at least t keys, give $c_i[x]$ an extra key by moving a key from x down into $c_i[x]$, moving a key from $c_i[x]$'s immediate left or right sibling up into x, and moving the appropriate child pointer from the sibling into $c_i[x]$.

b. If $c_i[x]$ and both of $c_i[x]$'s immediate siblings have t-1 keys, merge $c_i[x]$ with one sibling, which involves moving a key from x down into the new merged node to become the *median key* for that node.

Removal in B-Trees: Example

initial tree...

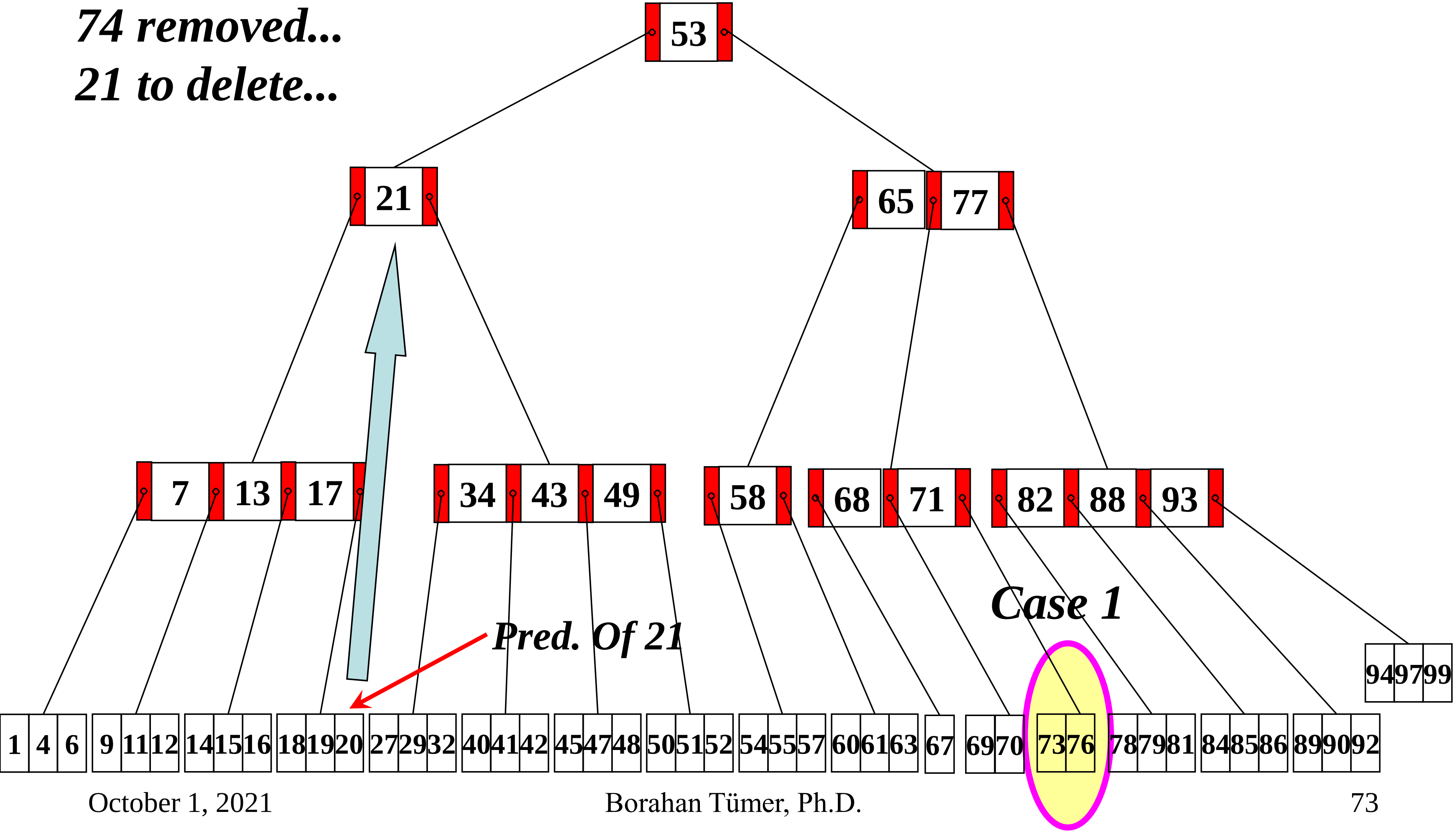
74 to delete...



Removal in B-Trees: Example

74 removed...

21 to delete...



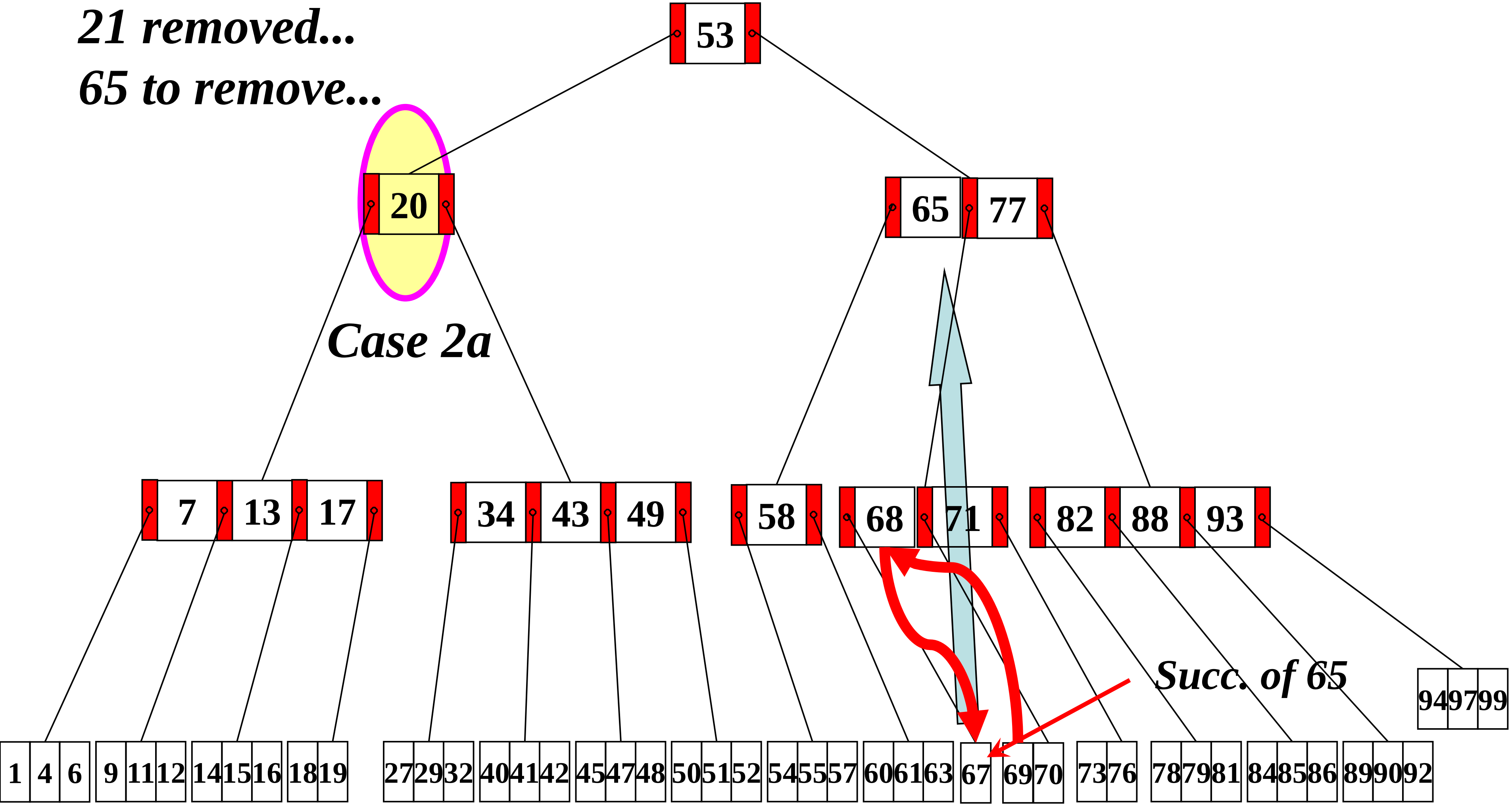
Pred. Of 21

Case 1

Removal in B-Trees: Example

21 removed...

65 to remove...



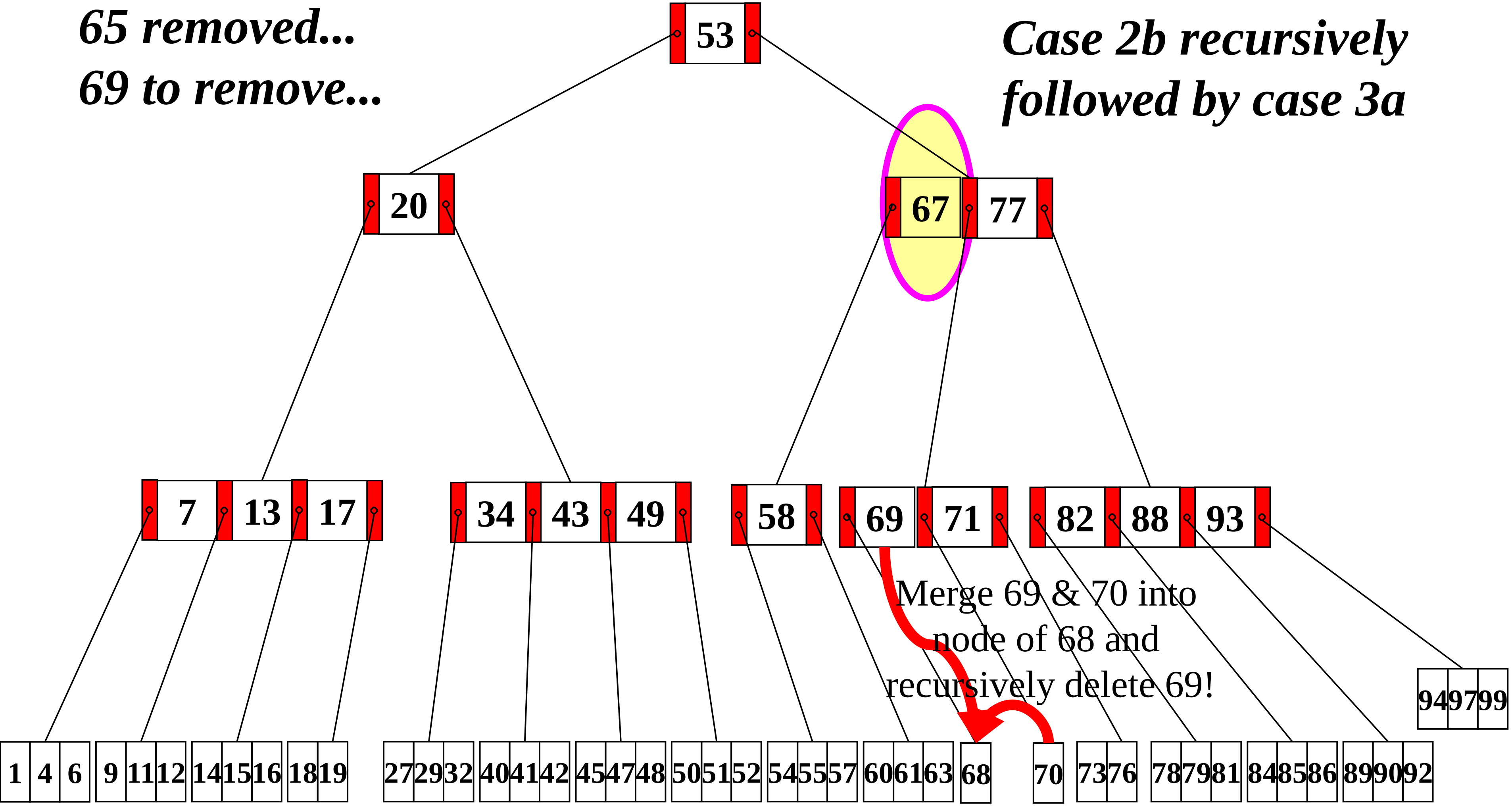
Case 2a

Succ. of 65

Removal in B-Trees: Example

65 removed...
69 to remove...

Case 2b recursively followed by case 3a

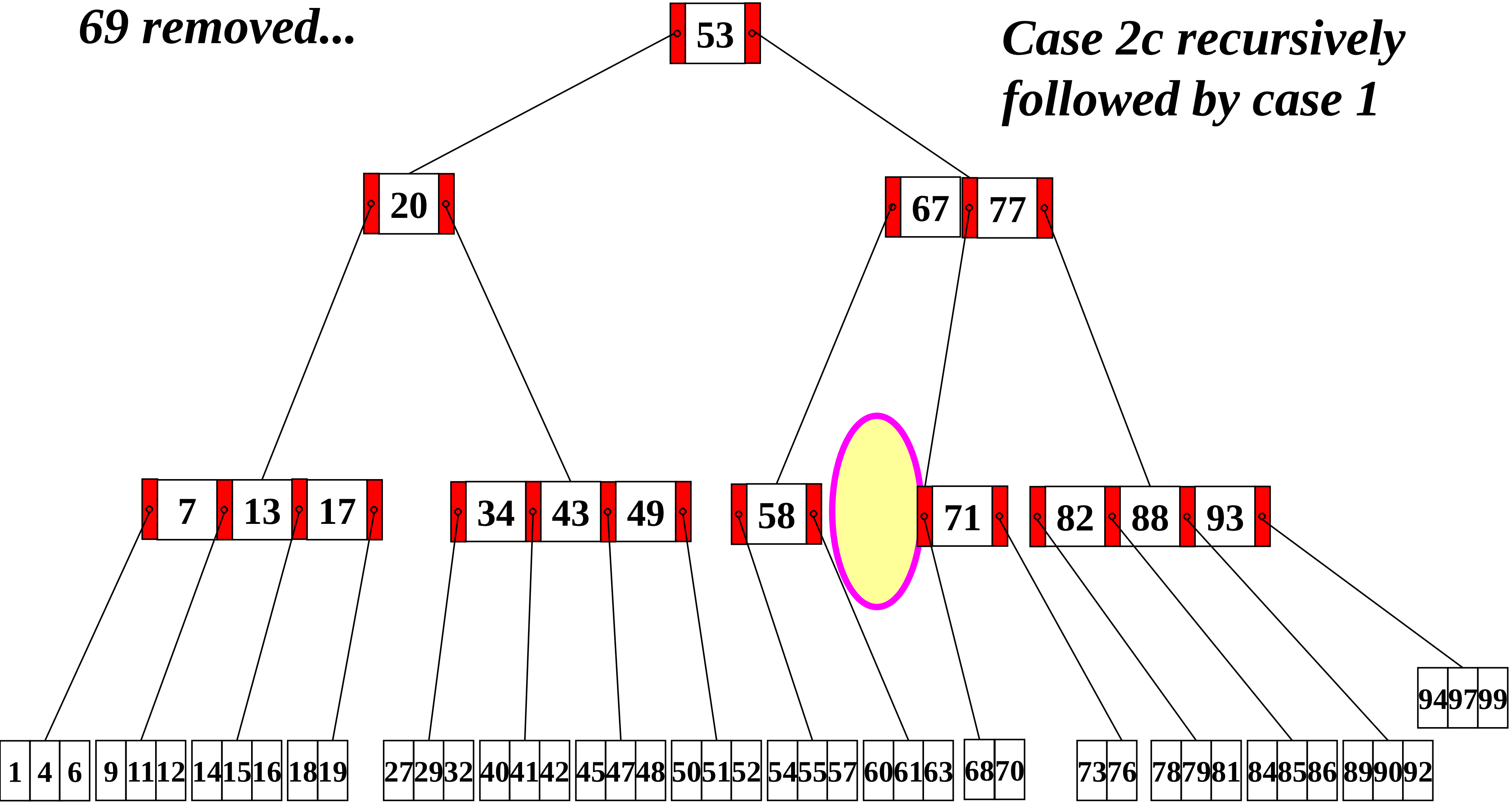


Merge 69 & 70 into node of 68 and recursively delete 69!

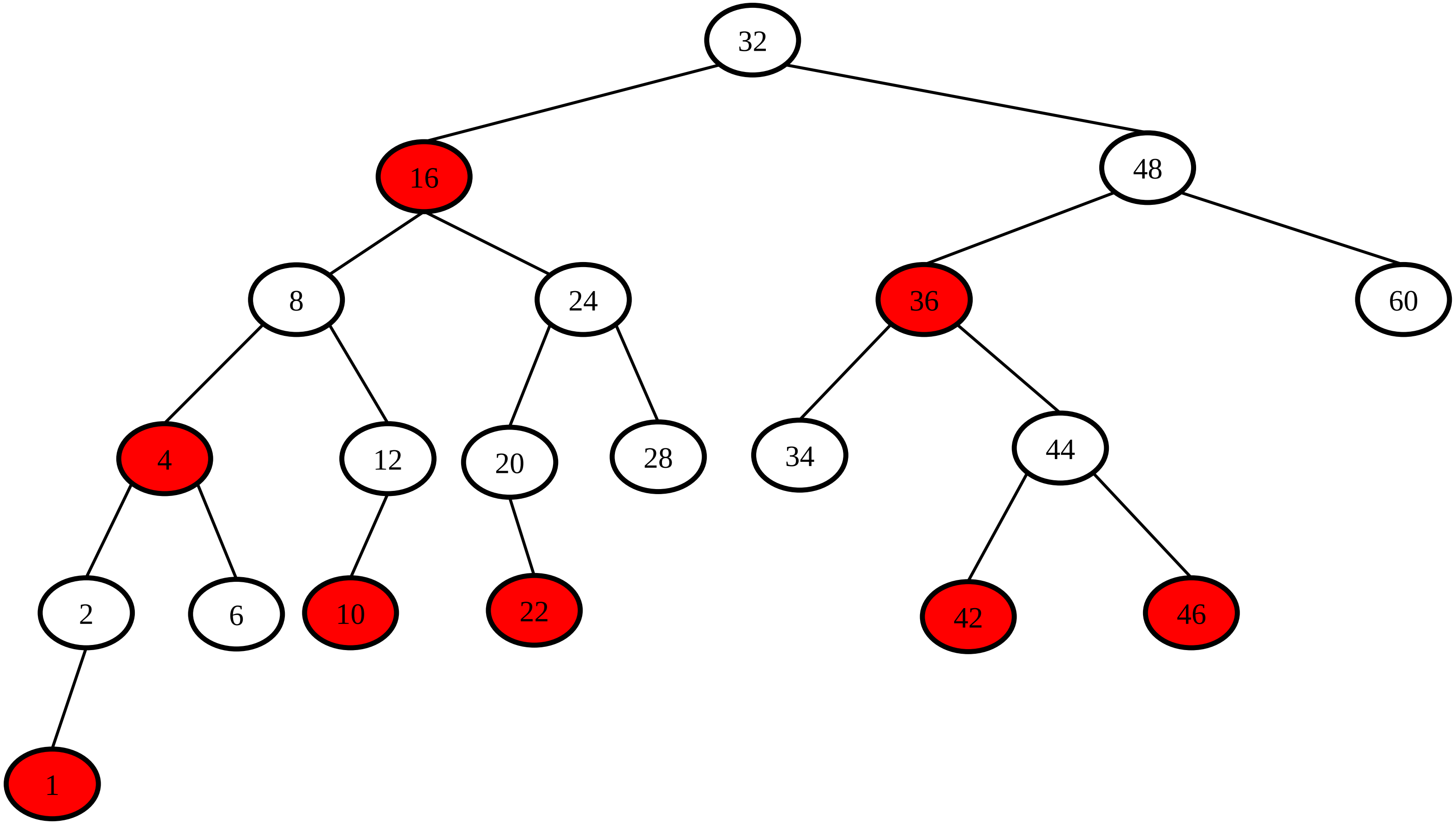
Removal in B-Trees: Example

69 removed...

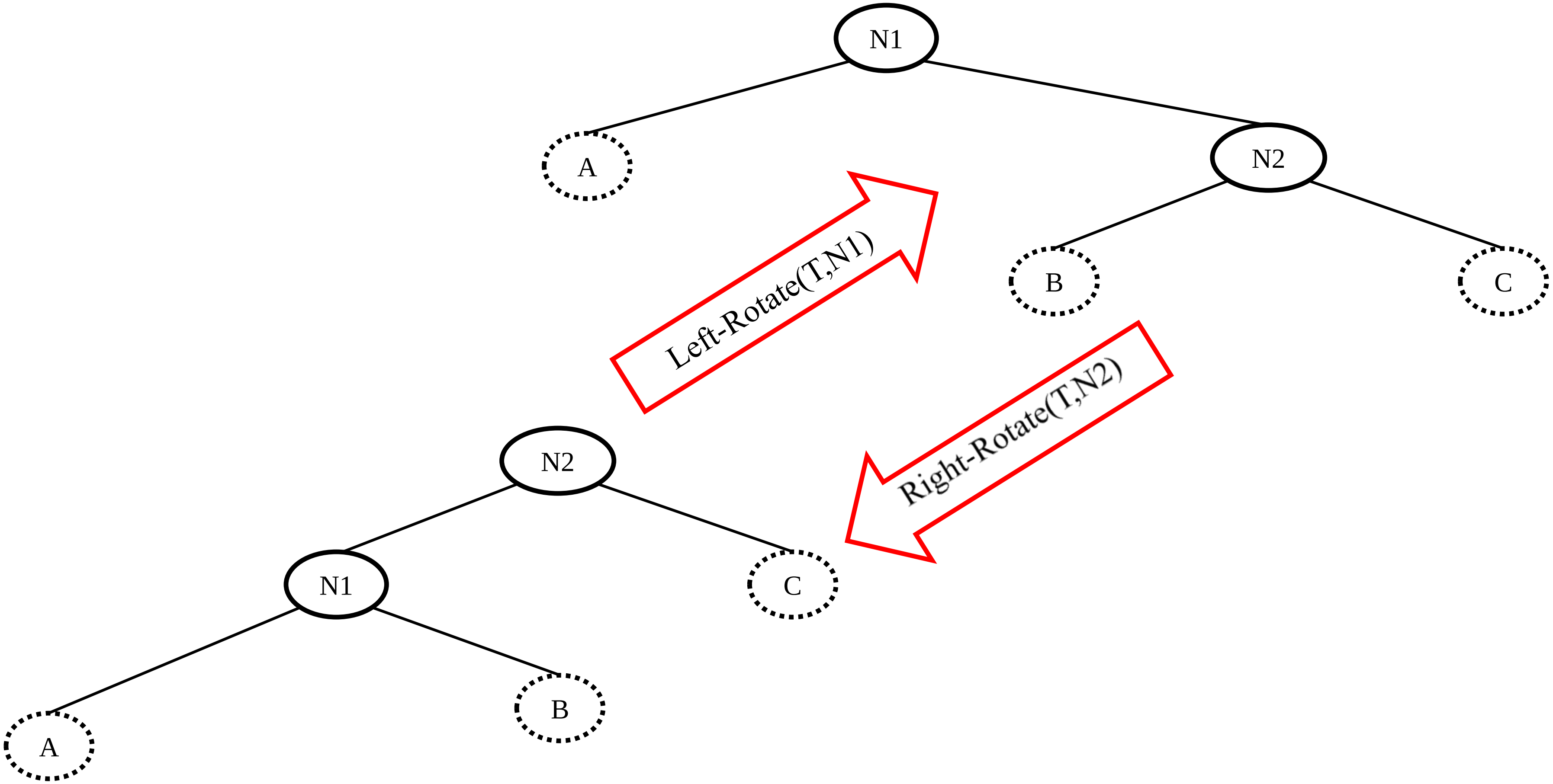
Case 2c recursively followed by case 1



Example RBT

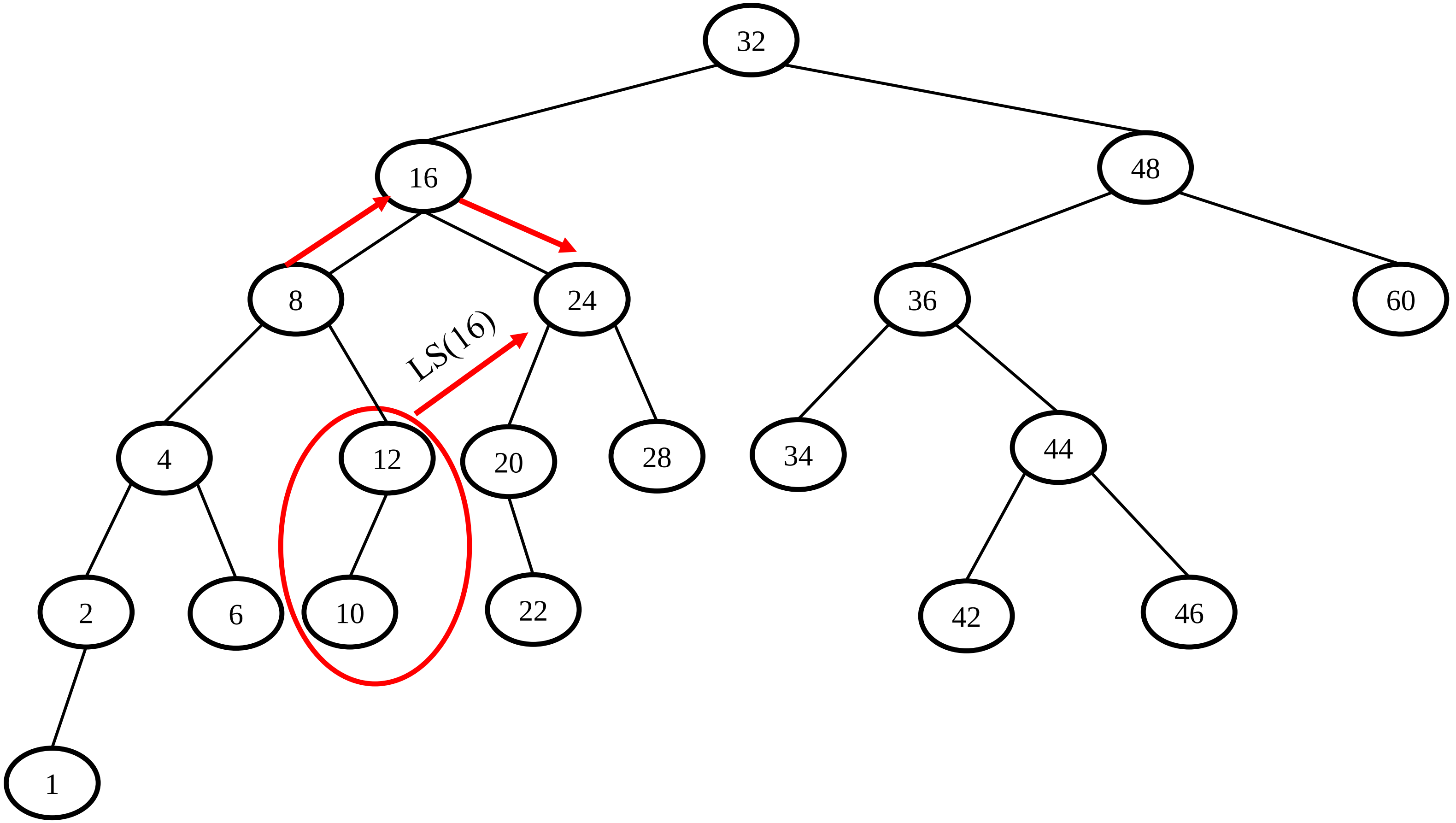


Rotations



Example RBT

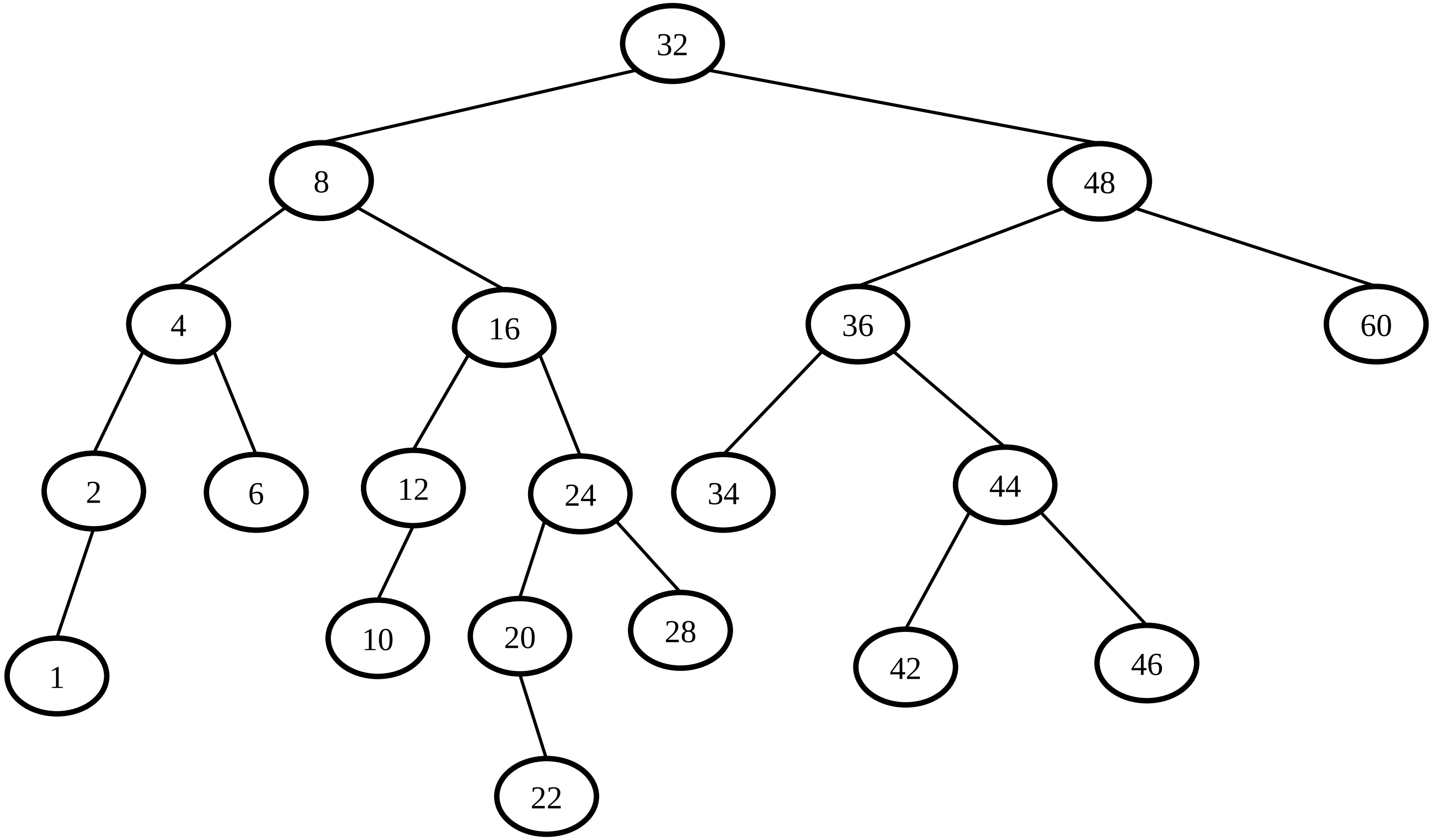
Right-Rotate(T,16)



LS stands for « Left Subtree of »

Example Rotation

Right-Rotate(T,16)



Insertion O(lgn)

```
RB-INSERT(T,z)
/*z inserted to T in O(logn)
y←nil[T]; x← root[T];
while x≠ nil[T] do
    y ←x
    if (key[z]<key[x])
        x ←left[x]
    else x ←right[x]
p[z]=y
if y=nil[T]
    root[T]←z
    else if (key[z]<key[y])
            left[y] ←z
        else right[y] ←z
left[z] ← nil[T]; right[z] ← nil[T];
color[z]← RED;
RB-INSERT-FIXUP(T,z)
```

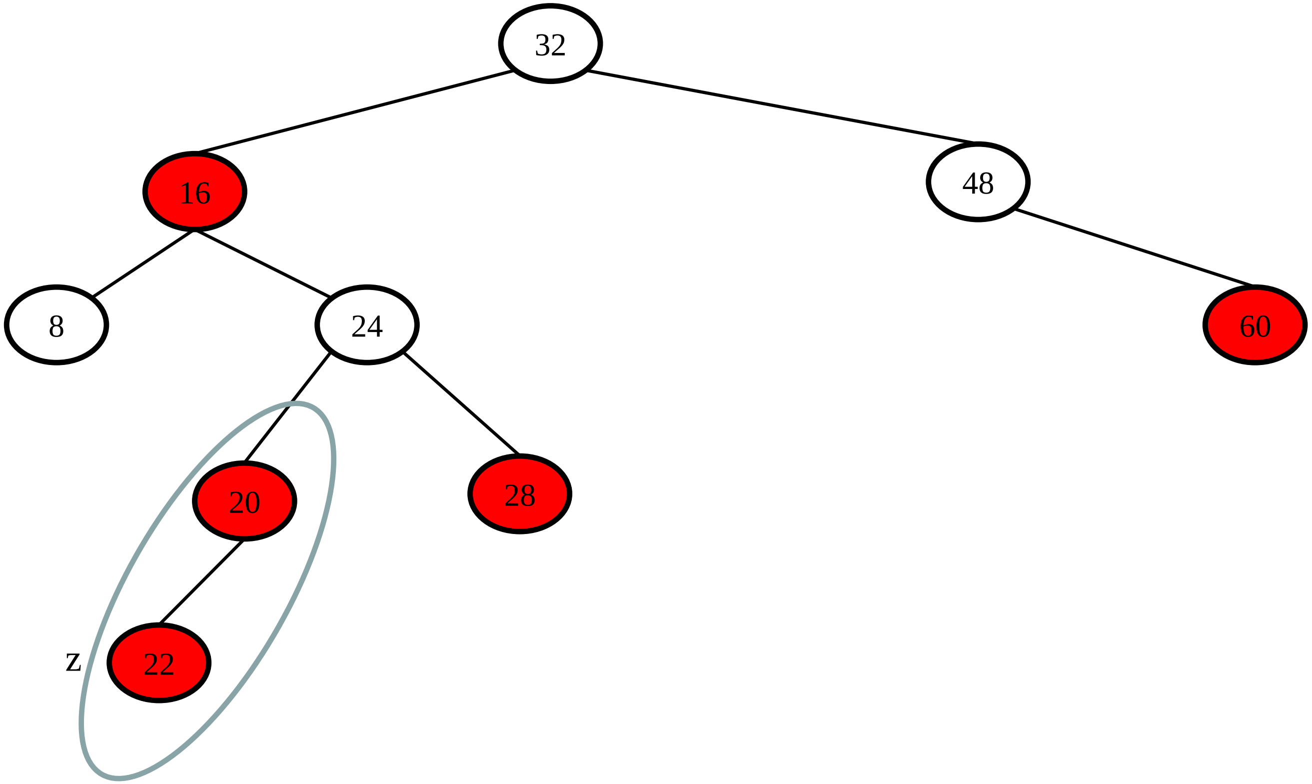
Fixing Up Colors after Insertion

```
•  RB-INSERT-FIXUP(T,z)
•  while color[p[z]] == RED do
•    if (p[z] == left[p[p[z]]])
         – y=right[p[p[z]]];
         – if (color[y]==RED)
                 •  color[p[z]]=BLACK
                 •  color[y]=BLACK
Case 1           •  color[p[p[z]]]=RED
                 •  z=p[p[z]]
         – else if (z==right[p[z]])
                       – z=p[z]
Case 2                 – LEFT-ROTATE(T,z)
                 •  color[p[z]]=BLACK
Case 3           •  color[p[p[z]]]=RED
                 •  RIGHT-ROTATE(T,p[p[z]])
```

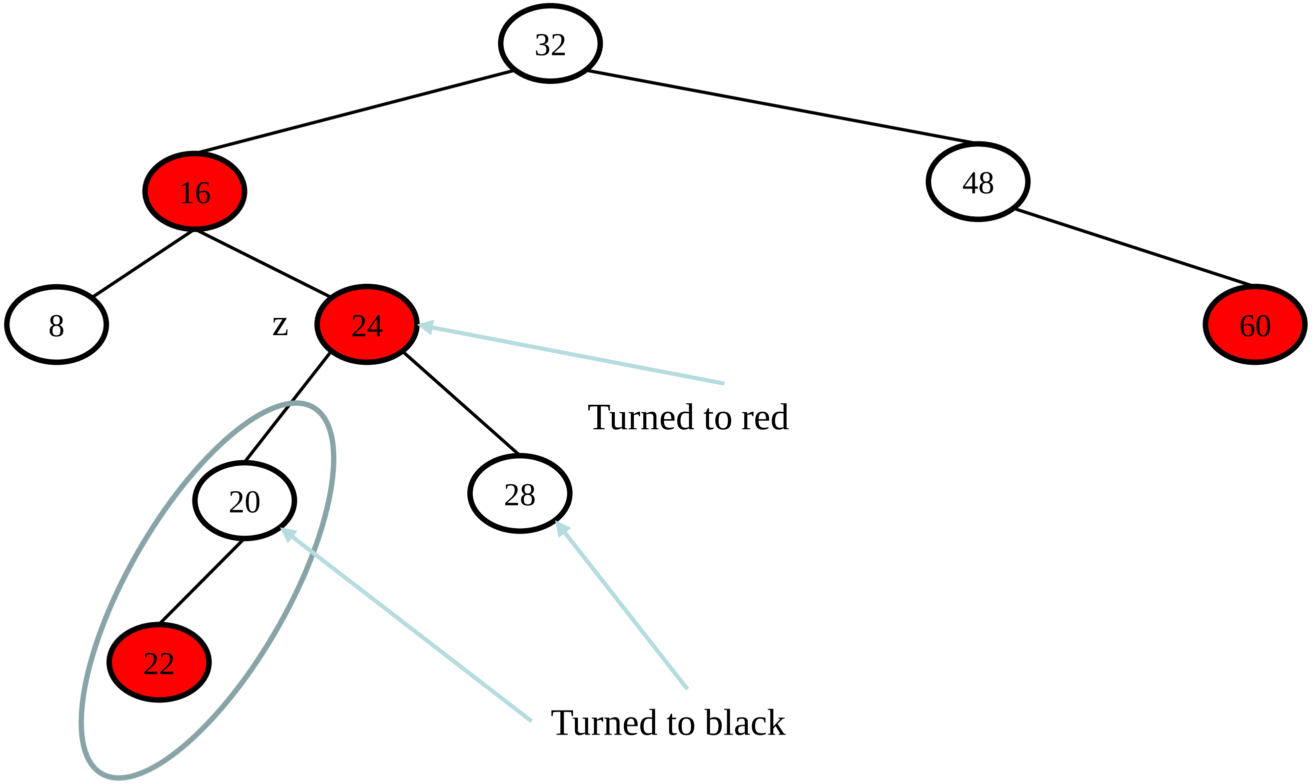
```
•  else    //** if (p[z] ≠ left[p[p[z]]])
         – y=left[p[p[z]]];
         – if (color[y]==RED)
                 •  color[p[z]]=BLACK
                 •  color[y]=BLACK
                 •  color[p[p[z]]]=RED              Case 1
                 •  z=p[p[z]]
         – else if (z==left[p[z]])
                       – z=p[z]                     Case 2
                       – RIGHT-ROTATE(T,z)
                 •  color[p[z]]=BLACK
                 •  color[p[p[z]]]=RED              Case 3
                 •  LEFT-ROTATE(T,p[p[z]])
•  color[root[T]]=BLACK;
```

Example: Case 1

Case 1: ***z***’s uncle y is ***red***.



Example: Case 1 solved

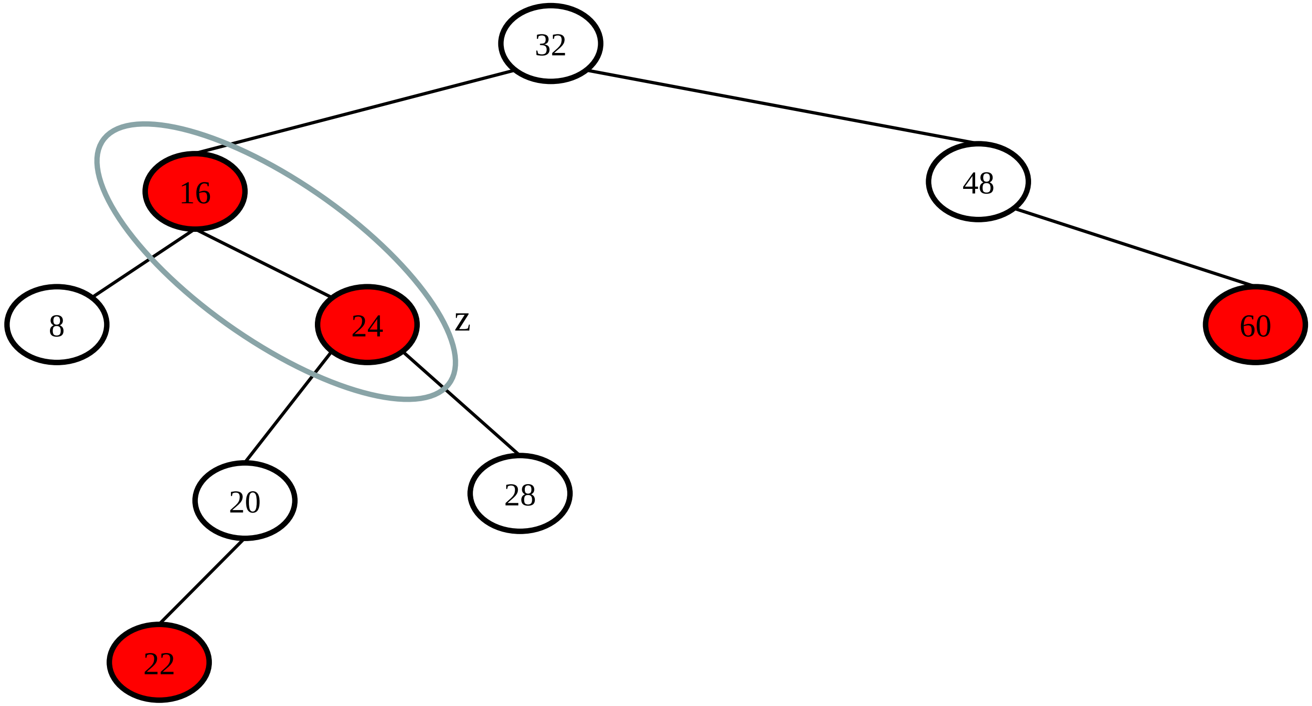


Turned to red

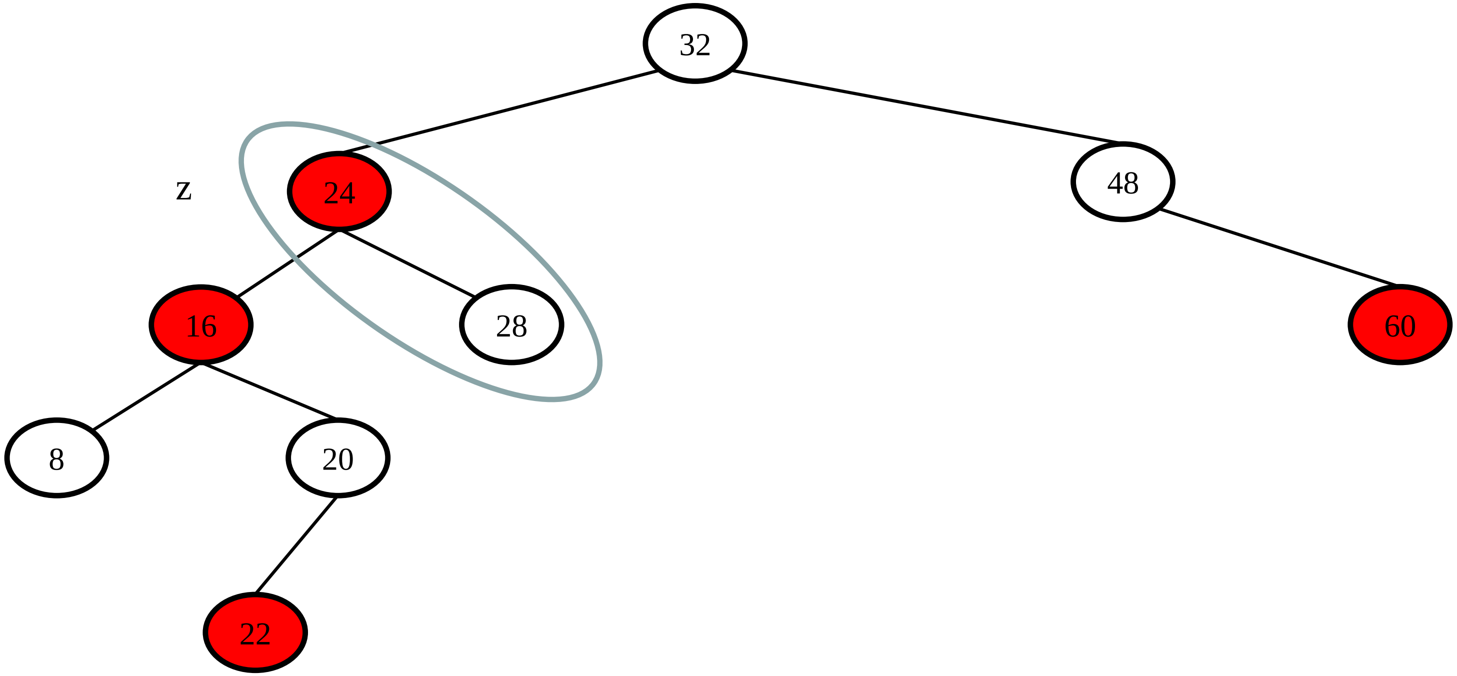
Turned to black

Example: Case 2

Case 2: ***z***’s uncle y is **black** and ***z*** is a right child

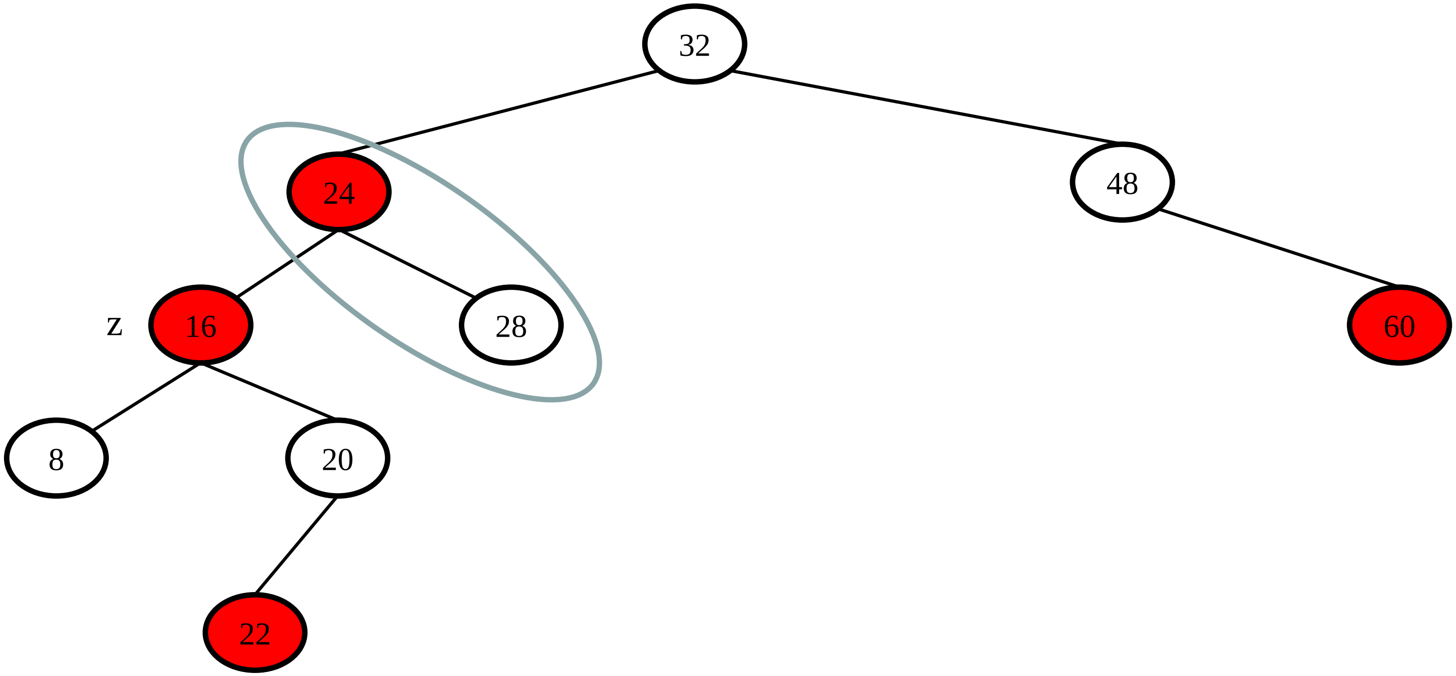


Example: Case 2 solved



Example: Case 3

Case 3: ***z***’s uncle y is **black** and ***z*** is a left child



Example: Case 3 solved