MAT1034 – MATEMATİK II / 27.03.2017 / VİZE SINAVI

CEVAPLAR

15 puan

1) $f(x)=\ln(-x^2+5x-4)$ fonksiyonun tanım kümesi $-x^2+5x-4>0 \Leftrightarrow x^2-5x+4<0$

$\Leftrightarrow x^2-5x+4=(x-4)(x-1)<0$ olmalıdır.

$\rightarrow$ i) $x<4$ ve $x>1 \Rightarrow \underline{1<x<4}$ olmalıdır

$\rightarrow$ ii) $x<1$ ve $x>4$ mümkün değil.

$D(f)=(1,4)$ olur.

* $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\ln(-(x-4)(x-1))=\ln(0)=-\infty$

ve $\lim\limits_{x\to 4} f(x)=\lim\limits_{x\to 4}\ln(-(x-4)(x-1))=\ln(0)=-\infty$

olduğundan $x=1$ ve $x=4$ doğruları dikey asimptottur.

* $\lim\limits_{x\to +\infty} f(x)$ durumu mevcut olmaz, çünkü $1<x<4$ için fonk. tanımlıdır. Yatay asimptot yoktur.

* $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}$, $x\to\infty$ olmadığından mevcut değildir. Ancak $\lim\limits_{x\to 4}\dfrac{f(x)}{x}=\dfrac{-\infty}{4}=-\infty$ olur

Eğik asimptot yoktur.

* I. türev, $f'(x)=\dfrac{-2x+5}{-x^2+5x-4}=\dfrac{-2x+5}{-(x-4)(x-1)} \Rightarrow f'(x)=0 \Leftrightarrow -2x+5=0 \Leftrightarrow \boxed{x=5/2}$

kritik nokta

. $\boxed{x=4 \text{ ve } x=1 \text{ tekil noktalar}}$

x	1	5/2	4
$f'(x)$	+		−
f	$\nearrow$	max nok.	$\downarrow$

$f(5/2)=\ln(9/4)$.

. $1<x<5/2$ için $-2x+5>0 \Rightarrow f'(x)=\dfrac{(+)}{-(-)(+)}>0$

. $5/2<x<4$ için $-2x+5<0 \Rightarrow f'(x)=\dfrac{(-)}{-(-)(+)}<0$

* II. türev,

$$f''(x)=\frac{-2(-x^2+5x-4)-(-2x+5)(-2x+5)}{(-x^2+5x-4)^2}=\frac{2x^2-10x+8-(4x^2+25-20x)}{(-x^2+5x-4)^2}$$

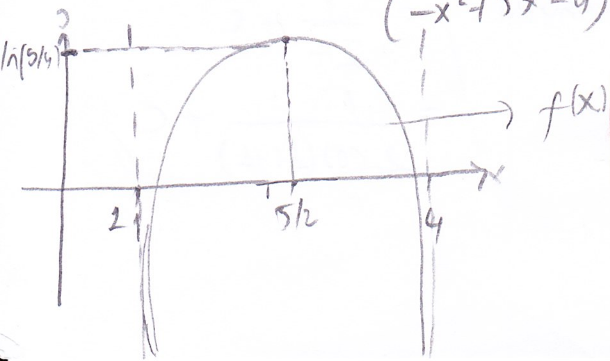
$$=\frac{-2x^2+10x-17}{(-x^2+5x-4)^2}$$

için $f''(x)=0 \Leftrightarrow 2x^2-10x+17=0$

$\Delta=10^2-4.2.17=100-136<0$

olduğundan reel kökü yoktur

Bu nedenle büküm noktası yoktur.



10 puan

2) $f(x)=(3-x^2)^2 \Rightarrow f'(x)=2(3-x^2).(-2x)=-4x(3-x^2)=4x(x^2-3)$

$$f'(x)=0 \Leftrightarrow \boxed{x=0,\ x=\mp\sqrt{3} \text{ kritik noktalar}}$$

$$f''(x)=4(x^2-3)+4x(2x)=4x^2-12+8x^2=12(x^2-1)$$

$$f''(x)=0 \Leftrightarrow x^2-1=0 \Leftrightarrow \boxed{x=\mp 1 \text{ büküm noktası olabilir}}$$

x		-1		$+1$	
$f''(x)$	$+$		$-$		$+$
	$\cup$		$\cap$		$\cup$

$f''(x)=12(x-1)(x+1)$

$x<-1 \Rightarrow f''=(-)(-)>0$

$-1<x<1 \Rightarrow f''=(-)(+)<0$

$x>1 \Rightarrow f''=(+)(+)>0$

f fonk $(-1,+1)$ aşağı içbükey, $(-\infty,-1)\cup(1,+\infty)$ aralıklarında yukarı içbükey olur

Ayrıca $f''(0)=12(-1)<0$ olduğundan $x=0$ yerel max. noktası. $\rightarrow x=0$ global max

$f''(\sqrt{3})=12(3-1)=24>0$ " $x=\sqrt{3}$ yerel min noktası

$f''(-\sqrt{3})=12(3-1)=24>0$ " $x=-\sqrt{3}$ " " $\Big\} \rightarrow x=\mp\sqrt{3}$ global min

Buradan $x=\mp 1$ noktaları büküm noktalarıdır//

3) 10 puan

$y''=x+\sin x,\ y(0)=2,\ y'(0)=0 \Rightarrow y=y(x)=?$

$$y'=\int y''\,dx=\int(x+\sin x)\,dx=\frac{x^2}{2}-\cos x+c_1 \Rightarrow y'(0)=0-\underbrace{\cos(0)}_{1}+c_1=0 \Rightarrow \boxed{c_1=1}$$

$$y=\int y'\,dx=\int\left(\frac{x^2}{2}-\cos x+c_1\right)dx=\frac{x^3}{6}-\sin x+c_1x+c_2$$

$$\Rightarrow y(0)=0-\underbrace{\sin(0)}_{0}+0+c_2=2 \qquad \boxed{c_2=2}$$

Böylece $\boxed{y(x)=\frac{x^3}{6}-\sin x+x+2}$

4) a) 7 puan

$$\int\frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt \underset{\substack{u=2t+1\\du=2dt}}{=} \frac{1}{2}\int\frac{\sin(u)}{\cos^2(u)}\,du \underset{\substack{t=\cos u\\dt=-\sin u\,du}}{=} \frac{1}{2}\int-\frac{dt}{t^2}=-\frac{1}{2}\,\frac{t^{-2+1}}{-1}+C$$

$$=\frac{1}{2t}+C$$

$$=\frac{1}{2\cos(2t+1)}+C\ //$$

b) 7puan

$$\int \frac{x}{\cos^2(x^2+1)}\,dx = \frac{1}{2}\int \frac{du}{\cos^2(u)} = \frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan(u)+C$$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$$= \frac{1}{2}\tan(x^2+1)+C \;//$$

c) 8puan

$$\int \underset{dv}{x^2}\,\underbrace{\arctan(x)}_{u}\,dx = \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

$v = x^3/3, \quad du = \dfrac{dx}{1+x^2}$

$$\rightarrow \int \frac{x^3}{1+x^2}\,dx = \int \frac{x^2\cdot x\,dx}{1+x^2} = \frac{1}{2}\int \frac{(t-1)}{t}\,dt = \frac{1}{2}\int\left(1-\frac{1}{t}\right)dt = \frac{1}{2}\left(t-\ln|t|\right)+C$$

$t = 1+x^2 \Rightarrow 2x\,dx = dt$

$$= \frac{1}{2}\left((1+x^2)-\ln(1+x^2)\right)+C \quad \text{veya} \quad \frac{1}{2}\left(x^2-\ln(1+x^2)\right)+C$$

$$\int x^2\arctan(x)\,dx = \frac{x^3}{3}\arctan x - \frac{1}{6}\left((1+x^2)-\ln(1+x^2)\right)+C \;//$$

d) 8puan

$$\int \underset{dv}{x}\,\underbrace{\ln(1+x^2)}_{u}\,dx = \frac{x^2}{2}\cdot\ln(1+x^2) - \frac{1}{\cancel{2}}\int x^2\cdot\frac{\cancel{2}x\,dx}{1+x^2} = \frac{x^2}{2}\ln(1+x^2) - \int \frac{x^3}{1+x^2}\,dx$$

$v = x^2/2$

$du = \dfrac{2x\,dx}{1+x^2}$

$$\rightarrow \int \frac{x^3}{1+x^2}\,dx = \int \frac{x^2\cdot x\,dx}{1+x^2} = \frac{1}{2}\int \frac{u\cdot du}{1+u} = \frac{1}{2}\int\left[1-\frac{1}{1+u}\right]du$$

$x^2 = u$

$2x\,dx = du$

$$= \frac{1}{2}\left[u-\ln|1+u|\right]+C = \frac{1}{2}\left[x^2-\ln(1+x^2)\right]+C.$$

$$\int x\cdot\ln(1+x^2)\,dx = \frac{x^2}{2}\ln(1+x^2) - \frac{1}{2}\left[x^2-\ln(1+x^2)\right]+C$$

$$= \frac{1}{2}\ln(1+x^2)\left[x^2+1\right] - \frac{x^2}{2}+C$$

5) a) 8 puan $\int \frac{dx}{x^3-4x^2+3x} = ?$ $\quad x^3-4x^2+3x = x(x^2-4x+3) = x(x-1)(x-3)$

$-3 \quad -1$

$$\frac{1}{x^3-4x^2+3} = \frac{1}{x(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3}$$

$$A = \lim_{x\to 0} \frac{1}{(x-1)(x-3)} = \frac{1}{3}, \quad B = \lim_{x\to 1} \frac{1}{x.(x-3)} = \frac{-1}{2}, \quad C = \lim_{x\to 3} \frac{1}{x(x-1)} = \frac{1}{3.2} = \frac{1}{6}$$

$$\rightarrow \int \frac{dx}{x^3-4x^2+3x} = \int \left(\frac{1}{3}\frac{1}{x} - \frac{1}{2}\frac{1}{x-1} + \frac{1}{6}\frac{1}{x-3}\right) dx = \frac{1}{3}\ln|x| - \frac{1}{2}\ln|x-1| + \frac{1}{6}\ln|x-3| + C \,//$$

5) b) 8 puan $\int \frac{dx}{(x^2-1)^2} = ?$ $\quad \frac{1}{(x^2-1)^2} = \frac{1}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{(x+1)} + \frac{E}{(x+1)^2}$

$$1 = A(x-1)(x+1)^2 + B(x+1)^2 + D(x+1)(x-1)^2 + E(x-1)^2$$

$x=-1 \Rightarrow \quad 1 = 0+0+0+E(-2)^2 \Rightarrow \boxed{E = 1/4}$

$x=1 \Rightarrow \quad 1 = 0 + B4 + 0 + 0 \Rightarrow \boxed{B = 1/4}$

Türev alırsak

$$0 = A\cdot(x+1)^2 + A(x-1)2(x+1) + B2(x+1) + D(x-1)^2 + D(x+1)2(x-1) + E2(x-1)$$

$x=-1 \Rightarrow \quad 0 = 0+0+0+D(-2)^2+0+E2(-2) \Rightarrow 4E = 4D \Rightarrow \boxed{E = D}$

$x=1 \Rightarrow \quad 0 = 4A + 0 + 4B + 0 + 0 + 0 \Rightarrow \boxed{A = -B}$

$$\Rightarrow \int \frac{dx}{(x^2-1)^2} = -\int \frac{1}{4}\frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{4}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\left(\ln|x+1| - \ln|x-1|\right) + \frac{1}{4}\left[\frac{-1}{x-1} - \frac{1}{x+1}\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{4}\frac{2x}{x^2-1} + C$$

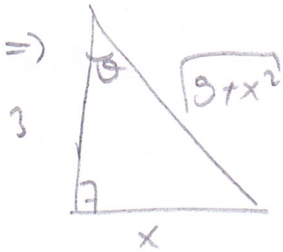
$$= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C \,//$$

5) c) 18 puan $\int \frac{dx}{(9+x^2)^{3/2}}$, $x=3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$

$9+x^2 = 9+9\tan^2\theta = 9\sec^2\theta$

$$\int \frac{dx}{(9+x^2)^{3/2}} = \int \frac{3\sec^2\theta\, d\theta}{(3^2\sec^2\theta)^{3/2}} = \int \frac{3\sec^2\theta\, d\theta}{3^3\sec^3\theta} = \frac{1}{9}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta\, d\theta = \frac{\sin\theta}{9}+C$$

$\to \tan\theta = x/3 \Rightarrow$



$\Rightarrow \sin(\theta) = \frac{x}{\sqrt{9+x^2}}$

$= \frac{x}{9(\sqrt{9+x^2})}+C$ //

d) 18 puan $I=\int \frac{\sqrt[3]{x}}{x(\sqrt{x}+\sqrt[3]{x})}\,dx$, $x=t^6 \Rightarrow dx = 6t^5dt$.

$$I=\int \frac{t^2\; 6t^5\, dt}{t^6(t^3+t^2)} = 6\int \frac{\cancel{t^7}\, dt}{\cancel{t^8}(t+2)} = 6\int \frac{dt}{t(t+1)} = 6\int \left[\frac{1}{t}-\frac{1}{t+1}\right]dt$$

$$= 6\left[\ln|t| - \ln|t+2|\right]+C = \ln|t^6| - \ln((1+t)^{1/6})+C$$

$$= \ln|x| - \ln((1+x^6)^{1/6})+C = \ln\left(\frac{x}{(1+x^6)^{1/6}}\right)+C$$

6) $\int \frac{e^x}{4+e^{2x}}\,dx$; $\frac{e^x}{4+e^{2x}} = \frac{e^x}{(e^x+2)^2-4e^x}$, $u=e^x+2 \Rightarrow e^x = u-2$

$du = e^x dx$

$$\int \frac{e^x}{4+e^{2x}}\,dx = \int \frac{e^x}{(e^x+2)^2-4e^x}\,dx = \int \frac{du}{u^2-4(u-2)} = \int \frac{du}{u^2-4u+8} = \int \frac{du}{(u-2)^2+4}$$

$$= \int \frac{du}{4\left(\underbrace{\left(\frac{u-2}{2}\right)}_{s}^2+1\right)} = \frac{1}{4}\int \frac{2\,ds}{s^2+1} = \frac{1}{2}\arctan(s)+C = \frac{1}{2}\arctan\left(\frac{u-2}{2}\right)+C$$

$\frac{du}{2} = ds$

$$= \frac{1}{2}\arctan\left(\frac{e^x}{2}\right)+C \;//$$

veya $e^x=u$ dönüşümü ile

$$\int \frac{du}{4+u^2} = \int \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta\, d\theta} = \frac{1}{2}\int d\theta = \frac{\theta}{2} = \frac{1}{2}\arctan\left(\frac{e^x}{2}\right)+C \quad (u=e^x)$$

$u = 2\tan\theta \Rightarrow \tan\theta = u/2 \Rightarrow \theta = \arctan(u/2)$

7) a) 6puan

$$\int_0^{16}\left(8y-\frac{2}{y^{1/4}}\right)dy = \frac{8y^2}{2}-2\cdot\frac{y^{-\frac{1}{4}+1}}{3/4}\Bigg|_{y=0}^{16=2^4}$$

$$= 4(16^2)-2\cdot\frac{4}{3}\cdot y^{3/4}\Big|_0^{2^4} = 4\cdot 16^2-\frac{8}{3}(2^3)$$

$$= 64\cdot 16-\frac{64}{3}$$

$$= 64\left(16-\frac{1}{3}\right)=64\cdot\frac{47}{3} \text{ //}$$

b) 10puan

$$|x-2| = \begin{cases} x-2, & x\geq 2 \\ -(x-2), & x<2 \end{cases}$$

$$\rightarrow \int_0^3 (x+|x-2|)\,dx = \int_0^3 x\,dx - \int_0^2 (x-2)\,dx + \int_2^3 (x-2)\,dx = \frac{x^2}{2}\Bigg|_{x=0}^{3} - \frac{(x-2)^2}{2}\Bigg|_{x=0}^{2} + \frac{(x-2)^2}{2}\Bigg|_{x=2}^{3}$$

$$= \frac{9}{2}-\left(0-\frac{4}{2}\right)+\left(\frac{1}{2}-0\right)$$

$$= \frac{9}{2}+\frac{4}{2}+\frac{1}{2} = 7 \text{ //}$$

$$\tilde{f} = \frac{1}{3}\int_0^3 f(x)\,dx = \frac{7}{3} \text{ //}$$