MATK034 - MATEMATİK II - # 04.04.2018 #

ARASINAV ÇÖZÜMLERİ

1) (4+6+5=15p)

$$f(x)=\frac{1}{\sqrt{4-x^2}} \quad (-2<x<2) \text{ fonk için}$$

a) $\lim_{x\to\mp 2} f(x) = \lim_{x\to\mp 2} \frac{1}{\sqrt{4-x^2}} = \frac{1}{0} = +\infty$ olduğundan $x=\mp 2$ doğruları düşey asimptot olur.

f fonk $-2<x<2$ için tanımlı olduğundan $\lim_{x\to\mp\infty} f(x)$ limitine bakılamaz!!! Bu nedenle yatay asimptotu mevcut değildir.

b) $f'(x) = -\frac{1}{2}(4-x^2)^{-3/2}(-2x) = \frac{x}{(4-x^2)^{3/2}} \Rightarrow x=0$ kritik nokta

$x=\mp 2$ tekil nokta

$$f''(x) = \frac{1.(4-x^2)^{3/2} - x.\,3/2\,(4-x^2)^{1/2}.(-2x)}{(4-x^2)^3} = \frac{(4-x^2)^{1/2}\left[(4-x^2)+3x^2\right]}{(4-x^2)^3}$$

$$\boxed{f''(x) = \frac{4+2x^2}{(4-x^2)^{5/2}}}$$

$f''(x)=0 \Leftrightarrow 4+2x^2=0 \Leftrightarrow x^2=-2$ olup, reel çözümü mevcut değildir.

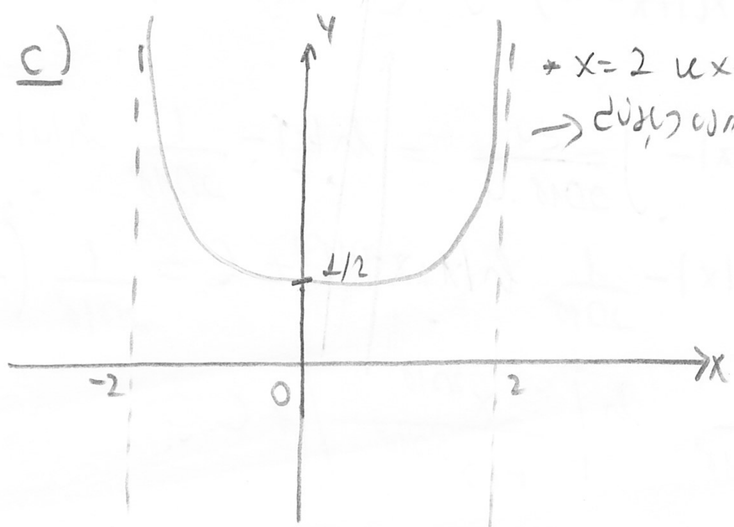
Ayrıca tanım kümesinde $\forall x\in(-2,2)$ için $f''(x) = \frac{\overbrace{4+2x^2}^{>0}}{\underbrace{(4-x^2)^3}_{>0}} > 0$ olduğundan f konveks fonk olur.

| x | -2 | 0 | 2 |
|---|---|---|---|
| f' | − | + | |
| f'' | + | + | |
| f | ∪↓ | ∪↗ | |

yerel min

$f(0) = \frac{1}{\sqrt{4}} = \frac{1}{2}$

c)




* $x=2$ ve $x=-2$ → düşey asimptot

2)(10p) $\forall x \in \mathbb{R}$ için $\displaystyle\int_{x^2+x}^{x^3-2} f(t)\,dt = x.\cos(\pi x) \Rightarrow f(6)=?$

Leibnitz integral formulünden

$$\frac{d}{dx}\left(\int_{x^2+x}^{x^3-2} f(t)\,dt\right) = \frac{d}{dx}\left(x.\cos(\pi x)\right)$$

$$3x^2 f(x^3-2) - (2x+1) f(x^2+x) = \cos(\pi x) - \pi x \sin(\pi x)$$

$x=2$ için $\quad 3.2^2 f(8-2) - 5.f(4+2) = \underbrace{\cos(2\pi)}_{1} - 2\pi \underbrace{\sin(2\pi)}_{0}$

$$12 f(6) - 5 f(6) = 1 \Rightarrow \boxed{f(6) = \frac{1}{7}}$$

3)(12p) $\displaystyle\int \frac{dx}{x(1+x^{2018})} = \int \frac{du}{2018x^{2017}.x.u} = \int \frac{du}{2018\,u(u-1)} = \frac{1}{2018}\int \frac{du}{u(u-1)}$

$u = 1+x^{2018} \Rightarrow du = 2018x^{2017}dx$

$dx = \dfrac{du}{2018x^{2017}}$ ve $x^{2018} = u-1$

$x^{2018} = u-1$

$$= \frac{1}{2018}\left[\int \frac{du}{u-1} - \int \frac{du}{u}\right] = \frac{1}{2018}\left[\ln|u-1| - \ln|u|\right] + C$$

$$= \frac{1}{2018}\ln\left|\frac{u-1}{u}\right| + C = \frac{1}{2018}\ln\left(\frac{x^{2018}}{1+x^{2018}}\right) + C$$

II. yol:

$$\int \frac{dx}{x(1+x^{2018})} = \int \left(\frac{1}{x} - \frac{x^{2017}}{1+x^{2018}}\right)dx = \ln|x| - \int \frac{x^{2017}}{1+x^{2018}}\,dx$$

$u = 1+x^{2018} \Rightarrow du = 2018x^{2017}dx$

$$= \ln|x| - \int \frac{du}{2018.u} = \ln|x| - \frac{1}{2018}\ln|u| + C$$

$$= \ln|x| - \frac{1}{2018}\ln|1+x^{2018}| + C = \frac{1}{2018}\left(2018\ln|x| - \ln|1+x^{2018}|\right) + C$$

$$= \frac{1}{2018}\ln\left|\frac{x^{2018}}{1+x^{2018}}\right| + C$$

4) (13p)

$$\int \frac{x+1}{x^2(x^2+1)}\,dx \quad \text{için} \quad \frac{x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Dx+E}{x^2+1}$$ biçiminde basit kesirlere ayrılır.

paydası eşitlenirse: $x+1 = A(x^2+1)x + B(x^2+1) + (Dx+E)x^2$

$$= x^3\underbrace{(A+D)}_{0} + x^2\underbrace{(B+E)}_{0} + x\underbrace{(A)}_{1} + \underbrace{B}_{1}$$

Böylece, $A=1$, $B=1$, $D=-A=-1$ ve $E=-B=-1$ elde edilir.

$$\int \frac{x+1}{x^2(x^2+1)}\,dx = \int \frac{dx}{x} + \int \frac{dx}{x^2} - \int \frac{x+1}{x^2+1}\,dx = \ln|x| + \frac{x^{-2+1}}{-1} - \underbrace{\int \frac{x\,dx}{x^2+1}}_{\substack{u=x^2+1 \\ du=2x\,dx}} - \int \frac{dx}{x^2+1}$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \arctan(x) + C.$$

5) (15p)

a) $$\int (1+\sin(2\theta))^2\,d\theta = \int \left[1 + 2\sin(2\theta) + \sin^2(2\theta)\right]d\theta = \theta + 2\cdot\frac{(-\cos(2\theta))}{2} + \int \frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= \theta - \cos(2\theta) + \frac{1}{2}\left(\int (1-\cos(4\theta))\,d\theta\right)$$

$$= \theta - \cos(2\theta) + \frac{\theta}{2} - \frac{1}{8}\sin(4\theta) + C$$

$$= \frac{3\theta}{2} - \cos(2\theta) - \frac{1}{8}\sin(4\theta) + C$$

b) $$\int \frac{d\theta}{\sin(\theta)} = \int \frac{\sin\theta}{\sin^2\theta}\,d\theta = \int \frac{\sin\theta}{1-\cos^2\theta}\,d\theta = \int \frac{-du}{1-u^2} = \int \frac{du}{u^2-1} = \int \frac{du}{(u-1)(u+1)}$$

$u=\cos\theta \Rightarrow du = -\sin\theta\,d\theta$

$$= \frac{1}{2}\int \left[\frac{1}{u-1} - \frac{1}{u+1}\right]du = \frac{1}{2}\left[\ln|u-1| - \ln|u+1|\right] + C$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right| + C \equiv \ln\left|\frac{\sin\theta}{\cos\theta+1}\right| + C$$

$$= -\ln|\cot\theta + \csc\theta| + C$$

6) (15p)

a) $$\int_0^1 \frac{x^2\,dx}{\sqrt{4-x^6}} = \int_0^1 \frac{dt/3}{\sqrt{4-t^2}} = \frac{1}{3}\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \frac{1}{3}\int_{\arcsin(0)}^{\arcsin(1/2)} \frac{\cancel{2\cos\theta}\,d\theta}{\cancel{2\cos\theta}}$$

$t=x^3 \Rightarrow 3x^2dx=dt$

$x=0 \Rightarrow t=0$

$x=1 \Rightarrow t=1$

$t=2\sin\theta \Rightarrow dt=2\cos\theta\,d\theta$

$\sqrt{4-t^2}=\sqrt{4(1-\sin^2\theta)}$

$=2\cos\theta$

$\theta=\arcsin(t/2)$

$$= \frac{1}{3}\int_{\arcsin(0)}^{\arcsin(1/2)} 1.\,d\theta = \frac{1}{3}\,\theta\Big|_{\arcsin(0)}^{\arcsin(1/2)} = \frac{1}{3}\Big[\underbrace{\arcsin(1/2)}_{\pi/6}-\underbrace{\arcsin(0)}_{0}\Big]$$

$$=\frac{\pi}{18}$$

b) $$\int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^t\,2t\,dt = 2\int_0^1 \underset{\downarrow\, u}{t}\,\underbrace{e^t\,dt}_{dv} = 2\Big[t.e^t\Big|_{t=0}^{1} - \int_0^1 e^t\,dt\Big]$$

$t=\sqrt{x} \Rightarrow t^2=x$

$2t\,dt=dx$

$x=0 \Rightarrow t=0$

$x=1 \Rightarrow t=1$

$du=dt$

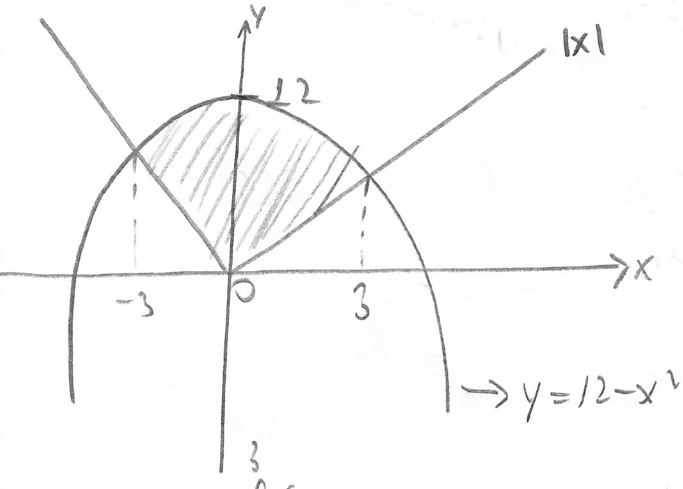
$v=e^t$

$$= 2\Big[(1.e^1-0)-e^t\Big|_{t=0}^{1}\Big] = 2\Big[e-(e^1-e^0)\Big] = 2(\cancel{e}-\cancel{e}+1)$$

$$=2$$

7) (5+5=10p) $y=|x|$ grafiğinin üstünde $y=12-x^2$ grafiğinin altında kalan bölgenin

a) $y=12-x^2 \Rightarrow y-12=-x^2$




Kesişim noktaları

- $x>0$ için $x=12-x^2 \Leftrightarrow x^2+x-12=0$
  $(x+4)(x-3)=0$
  $\boxed{x=3}$
- $x<0$ için $-x=12-x^2 \Leftrightarrow x^2-x-12=0$
  $(x-4)(x+3)=0$
  $\boxed{x=-3}$

b) $$\text{Alan}=\int_{-3}^{0}[12-x^2-(-x)]\,dx+\int_0^3(12-x^2-x)\,dx$$

veya $$\text{Alan}=\int_{-3}^{3}[12-x^2-|x|]\,dx$$ veya $$\text{Alan}=2\int_0^3(12-x^2-x)\,dx=2\int_{-3}^{0}(12-x^2+x)\,dx$$

8) $(2+8=10p)$ $\int_0^3 \frac{dx}{\sqrt{6-2x}}$ integrali

a) $f(x)=\frac{1}{\sqrt{6-2x}}$ fonk. $x=3$ noktasında sonsuz süreksizliğe sahip olduğundan bu integral 2. tip Genelleştirilmiş İntegral'dir

b) $$\int_0^3 \frac{dx}{\sqrt{6-2x}} = \lim_{R\to 3^-} \int_0^R \frac{dx}{\sqrt{6-2x}} = \lim_{R\to 3^-} \int_6^{6-2R} \frac{-du/2}{\sqrt{u}} = \lim_{R\to 3^-} \frac{1}{2}\int_{6-2R}^{6} u^{-1/2}\,du$$

$$\left| \begin{array}{l} u=6-2x \Rightarrow du=-2dx \\ x=0 \Rightarrow u=6 \\ x=R \Rightarrow u=6-2R \end{array} \right|$$

$$= \frac{1}{2}\lim_{R\to 3^-} \left.\frac{u^{1/2}}{1/2}\right|_{u=6-2R}^{6} = \frac{1}{2}\lim_{R\to 3^-} 2\left[\sqrt{6}-\sqrt{6-2R}\right]$$

$$= \sqrt{6} - \underbrace{\lim_{R\to 3^-}\sqrt{6-2R}}_{0} = \sqrt{6} < \infty$$ olduğundan verilen integral yakınsaktır