MAT 1034 – MATEMATİK II. – BÜTÜNLEME CEVAP ANAHTARI

16.06.2017

1) (10 puan) $\int_{\sqrt{x}}^{x} f(t)\,dt = x.\sin(\pi x) \Rightarrow f(1)=?$

Her iki tarafın türevini alalım $\frac{d}{dx}\left(\int_{\sqrt{x}}^{x} f(t)\,dt\right) = \frac{d}{dx}\left(x.\sin(\pi x)\right)$

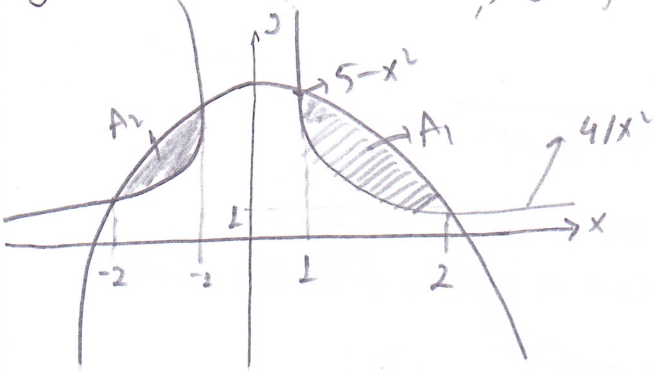
$\Rightarrow (x)'.f(x) - (\sqrt{x})'.f(\sqrt{x}) = \sin(\pi x) + x.\pi.\cos(\pi x)$

$f(x) - \frac{1}{2\sqrt{x}} f(\sqrt{x}) = \sin(\pi x) + \pi x.\cos(\pi x)$ elde ederiz.

$x=1 \Rightarrow f(1) - \frac{1}{2} f(1) = \underbrace{\sin(\pi)}_{0} + \pi.\underbrace{\cos(\pi)}_{-1} \Rightarrow \frac{1}{2} f(1) = -\pi \Rightarrow \boxed{f(1) = -2\pi}$

2) (12 puan) $y = 5-x^2$ ile $y = \frac{4}{x^2}$ eğrilerinin sınırladığı toplam alanı bulunuz.

$y-5 = -x^2 \Rightarrow$ kolları aşağıya doğru olan, tepe noktası $(0,5)$ olan paraboldür.



Bu iki eğrinin kesişim noktaları

$5-x^2 = \frac{4}{x^2} \Leftrightarrow 5x^2-(x^2)^2 = 4$

$x^2 = u \Rightarrow 5u - u^2 = 4$

$u^2 - 5u + 4 = 0 \Rightarrow (u-4)(u-1) = 0$

$u=4,\ u=1$

$u=4 \Rightarrow x^2 = 4 \Rightarrow x = \mp 2$

$u=1 \Rightarrow x^2 = 1 \Rightarrow x = \mp 1$ kesişim noktaları

İstenilen alan bu iki parça alanın toplamı olup, bu alanlar eşittir.

I. bölgedeki alanı bulalım $A_1 = \int_1^2 \left(5-x^2-\frac{4}{x^2}\right)dx = 5x - \frac{x^3}{3} - \frac{4.x^{-1}}{-1}\Big|_{x=1}^{2}$

$= 5.(2-1) - \frac{1}{3}(2^3-1^3) + 4\left(\frac{1}{2}-1\right)$

$= 5 - \frac{7}{3} - 2 = 3 - \frac{7}{3} = \frac{2}{3}$ //

Böylece istenilen alan $\boxed{A = 2A_1 = \frac{4}{3}}$ bulunur.

3) ~~kopu~~ (8 puan)

$r = \sin(\theta) + \cos(\theta) \Rightarrow r^2 = \underbrace{r\cdot\sin\theta}_{x} + \underbrace{r\cdot\cos\theta}_{y} \Leftrightarrow x^2+y^2 = x+y$

$\Rightarrow (x^2-x)+(y^2-y)=0 \Leftrightarrow \left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}=0$

$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2} \Rightarrow$ merkezi $\left(\frac{1}{2},\frac{1}{2}\right)$ ve yarıçapı $\frac{1}{\sqrt{2}}$ olan çember.

4) (10 puan) $x = t^2\sin t$, $y = t^2\cos t$, $0 \le t \le 2\pi$ parametrik denklemi ile verilen eğri

$\frac{dx}{dt} = 2t\cdot\sin t + t^2\cos t$, $\frac{dy}{dt} = 2t\cos t - t^2\sin t$

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 = (2t\sin t + t^2\cos t)^2 + (2t\cos t - t^2\sin t)^2$

$= (4t^2\sin^2t + t^4\cos^2t + \cancel{4t^3\sin t\cos t}) + (4t^2\cos^2t + t^4\sin^2t - \cancel{4t^3\sin t\cos t})$

$= 4t^2(\sin^2t+\cos^2t) + t^4(\cos^2t+\sin^2t) = 4t^2+t^4$

Eğrinin uzunluğu

$L = \int_0^{2\pi}\sqrt{(x')^2+(y')^2}\,dt = \int_0^{2\pi}\sqrt{4t^2+t^4}\,dt = \int_0^{2\pi} t\sqrt{4+t^2}\,dt$

$u = 4+t^2 \Rightarrow du = 2t\,dt$

$t=0 \Rightarrow u=4$, $t=2\pi \Rightarrow u = 4+4\pi^2$

$= \int_4^{4(1+\pi^2)}\sqrt{u}\,\frac{du}{2} = \frac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_{u=4}^{4(1+\pi^2)} = \frac{1}{3}\left[[4(1+\pi^2)]^{3/2} - \underbrace{4^{3/2}}_{(2^2)^{3/2}}\right]$

$= \frac{8}{3}\left[(1+\pi^2)^{3/2}-1\right]$

5) a) (10 puan)

$\int \frac{e^t}{(e^{2t}+9)^{3/2}}dt = \int \frac{du}{(u^2+9)^{3/2}} = \int \frac{3\sec^2\theta\,d\theta}{(3^2\sec^2\theta)^{3/2}} = \int \frac{\cancel{3}\sec^{\cancel{2}}\theta\,d\theta}{3^{\cancel{3}2}\sec^{\cancel{3}}\theta}$

$e^t = u \Rightarrow du = e^t dt$

$u = 3\tan\theta$

$u^2+9 = 9(1+\tan^2\theta) = 9\sec^2\theta$

$du = 3\sec^2\theta\,d\theta$

$\theta = \arctan(u/3) = \arctan(e^t/3)$

$= \int \frac{d\theta}{9\sec\theta} = \frac{1}{9}\int\cos\theta\,d\theta$

$= \frac{1}{9}\sin\theta + C = \frac{1}{9}\sin\left(\arctan\left(\frac{e^t}{3}\right)\right)+C$

5) b) (10 puan)

$$\int_0^1 \frac{x \ln(\sqrt{1+x^2})}{1+x^2}\,dx = \int_1^{\sqrt{2}} \frac{\ln u}{u^2}\, u\,du = \int_1^{\sqrt{2}} \frac{\ln u}{u}\,du = \int_0^{\ln\sqrt{2}} t\,dt = \frac{t^2}{2}$$

$\ln u = t \Rightarrow \frac{du}{u} = dt$

$$= \frac{(\ln\sqrt{2})^2}{2} \quad //$$

$u = \sqrt{1+x^2} \Rightarrow u^2 = 1+x^2$

$2u\,du = 2x\,dx \Rightarrow x\,dx = u\,du$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=\sqrt{2}$

veya $u = \ln(\sqrt{1+x^2})$

$$du = \frac{\frac{2x}{2\sqrt{1+x^2}}}{(\sqrt{1+x^2})} = \frac{x}{1+x^2}\,dx \qquad \int_0^{\ln\sqrt{2}} u\,du$$

c) (10 puan)

$$\int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{u \cdot 2u\,du}{u^2+1} = \int \frac{2u^2\,du}{u^2+1} = \int \frac{2(u^2+1)-2}{u^2+1}\,du$$

$\sqrt{x} = u \Rightarrow u^2 = x \Rightarrow 2u\,du = dx$

$$= 2\int du - 2\int \frac{du}{u^2+1} = 2u - 2\arctan(u) + C$$

$$= 2(\sqrt{x} - \arctan(\sqrt{x})) + C \quad //$$

d) (10 puan)

$$\int \sec^3 x \cdot \tan^5 x\,dx = \int \underbrace{\sec^2 x}_{u^2} \cdot \underbrace{\tan^4 x}_{(u^2-1)^2} \cdot \underbrace{\sec x \cdot \tan x\,dx}_{du}$$

$1+\tan^2 x = \sec^2 x$

$u = \sec x \Rightarrow du = \sec x \cdot \tan x\,dx$

$(\tan^2 x)^2 = (\sec^2 x - 1)^2 = (u^2-1)^2$

$$= \int u^2(u^2-1)^2\,du = \int u^2(u^4-2u^2+1)\,du = \int (u^6-2u^4+u^2)\,du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C = \frac{(\sec x)^7}{7} - \frac{2(\sec x)^5}{5} + \frac{\sec^3 x}{3} + C \quad //$$

6)(10 puan)

$\int_{-\infty}^{0} x\cdot e^x dx$ I. tip genelleştirilmiş integraldir

$$\int_{-\infty}^{0} x\cdot e^x dx = \lim_{c\to-\infty}\int_c^0 \underset{u}{x}\cdot \underbrace{e^x dx}_{dv} = \lim_{c\to-\infty}\left[x\cdot e^x\Big|_c^0 - \int_c^0 e^x dx\right]$$

$v=e^x$

$$= \lim_{c\to-\infty}\left[0 - c\cdot e^c - e^x\Big|_c^0\right] = \lim_{c\to-\infty}\left[-c\cdot e^c-(e^0-e^c)\right]$$

$$= \lim_{c\to-\infty}\left[-c\cdot e^c - 1 + e^c\right] = -1 - \underbrace{\lim_{c\to-\infty} c\cdot e^c}_{} + \underbrace{\lim_{c\to-\infty} e^c}_{0}$$

$$= -1 - \lim_{c\to\infty}\frac{c}{e^{-c}}\left(\frac{\infty}{\infty}\right) = -1 - \lim_{c\to-\infty}\frac{1}{-e^{-c}} = -1 + \underbrace{\lim_{c\to-\infty} e^c}_{0} = -1 \;//$$

$\int_{-\infty}^{0} x\cdot e^x dx = -1 < \infty$ olduğundan integral yakınsaktır //

7)(10 puan). $\sum_{n=2}^{\infty}\frac{(-5)^n}{8^{2n}} = \sum_{n=2}^{\infty}\left(\frac{-5}{8^2}\right)^n$ serisinde $r=\left(\frac{-5}{8^2}\right)$ olup, $|r|<1$ olduğundan bu geometrik seri yakınsaktır. Toplamını bulalım

$$\sum_{n=2}^{\infty}\left(\frac{-5}{8^2}\right)^n = \left(-\frac{5}{8^2}\right)^2 + \left(\frac{-5}{8^2}\right)^3 + \cdots = \left(-\frac{5}{8^2}\right)^2\left[1 + \left(\frac{-5}{8^2}\right) + \left(-\frac{5}{8^2}\right)^2 + \cdots\right]$$

$$= \frac{25}{8^4}\cdot\frac{1}{1-\left(\frac{-5}{8^2}\right)} = \frac{25}{\cancel{8^4}\,8^2}\cdot\frac{\cancel{8^2}}{8^2+5} = \frac{25}{64\cdot 69}$$

8) a)(4 puan) $a_n = \left(\frac{n-3}{n}\right)^n = \left(1-\frac{3}{n}\right)^n$ olup, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1-\frac{3}{n}\right)^n = e^{-3}$

olup, bu dizi yakınsaktır

b)(6 puan) $a_n=\sqrt{n+1}-\sqrt{n} \Rightarrow \lim_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} =$

$=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ olup, bu dizi yakınsaktır.

Ayrıca, $f(x)=\sqrt{x+1}-\sqrt{x}$, $x\in\mathbb{N}$ için $f(n)=a_n$ olur. $f'(x)=\frac{1}{2}\left(\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}\cdot\sqrt{x+1}}\right) < 0$ Azalandır //