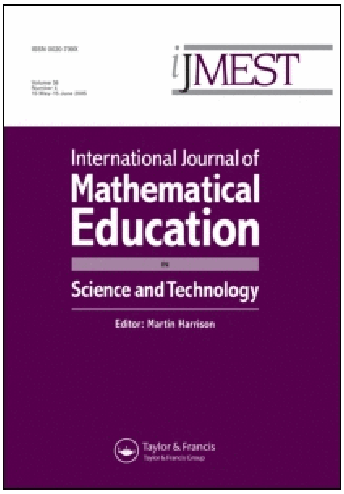

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Pre-service mathematics teachers' concept images of radian

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Pre-service mathematics teachers' concept images of radian

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This study investigates pre-service mathematics teachers' concept images of radian and possible sources of such images. A multiple-case study was conducted for this study. Forty-two pre-service mathematics teachers completed a questionnaire, which aims to assess their understanding of radian. Six of them were selected for individual interviews on the basis of theoretical sampling. The data indicated that participants' concept images of radian were dominated by their concept images of degree. As the data in this study suggested, pre-service mathematics teachers were reluctant to accept trigonometric functions with the inputs of real numbers but rather they use value in degrees. More interestingly, they have two distinct images of $\pi : \pi$ as an angle in radian and π as an irrational number.

Keywords: trigonometry; radian; concept image; pre-service mathematics teachers

1. Introduction

Trigonometry is an important topic in school mathematics. Using sine and cosine functions, it is possible to represent all periodic phenomena (such as earthquakes, heart rhythms, magnetic fields) as trigonometric functions as stated by Wu (2002) in Fi [1]. Concepts in trigonometry are related to both other concepts in mathematics and topics in science. Despite its importance, trigonometry is one of the topics in mathematics education research, which did not receive enough attention. As stated by Fi [1], much of the literature on trigonometry has focused on trigonometric functions (Even, 1989; Even, 1990; Bolte, 1993; Howald, 1998). Other studies focused on the learning and teaching of trigonometry and trigonometric functions with computers and calculators [2–5].

There is little research on teachers' understanding of trigonometry [1,6]. In his doctoral dissertation, Fi [1] investigated pre-service mathematics teachers' subject matter knowledge and pedagogical content knowledge of trigonometry. His study is the most comprehensive study, which directly focuses on this topic. He explored the subject knowledge of trigonometry considering the following components: definitions and terminology; degree and radian measures; co-functions; angles of rotation, coterminal angles and reference angles; special angles, their triangles, and their use to simplify computation; trigonometric functions and their graphs; domain and range; transformation

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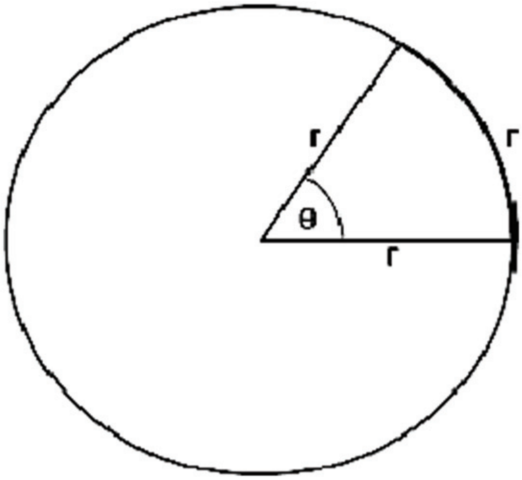

Figure 1. Visual explanation of 1 radian.

of trigonometric functions; even and odd functions; geometric underpinnings of trigonometry, for example, triangles.

This study focuses on pre-service mathematics teachers' understanding of a specific concept in trigonometry, namely the radian. Radian is an angle measure related to the arc length of a circle. For a circle, the central angle, which corresponds to the arc length that is equal to radius of the circle, is defined as 1 radian (Figure 1).

The radian of an angle can be described as the ratio of two lengths: the length of the arc of a central angle of a circle and the radius of the circle. In a unit circle, the radian of an angle is the arc of that central angle, since the radius is 1. Therefore, a radian angle is expressed as a real number such as $\sin 30$ while angles in degrees are expressed with degree notation such as $\sin 30^\circ$. Considering the wrapping metaphor (wrapping the real number line onto the unit circle), we can assign every real number to an angle in radian. This allowed mathematicians to define trigonometric functions with domain and range consisting of real numbers using the radian measure. However, angles in degrees are defined by dividing a circle into 360 segments and hence, cannot be used to define trigonometric functions.

Although the use of degree measure goes back to Babylonians, the use of radian is quite recent [7]. Thomas Muir and/or James Thompson were the first to use the word 'radian' in about 1870. However, mathematicians had been measuring angles in that way for a long time. For instance, Leonhard Euler (1707–1783) measured angles as the length of the arc cut-off in the unit circle in his Elements of Algebra. This way, he could construct his famous formula '$e^{it} = \cos t + i \sin t$'. In this formula, t is the length of an arc, which was later called the radian measurement of an angle [8].

The most detailed findings on the concept of radian in mathematics education literature were reported by Fi [1]. He found that although pre-service mathematics teachers were successful with converting between radians and degrees, none of them was able to accurately define the radian measure as a ratio of two lengths: the length of the arc of a central angle of a circle and the radius of the circle. He also found that pre-service teachers were more comfortable with degree measure than radian measure. Although they can move easily between degrees and radians, they do not have a deep understanding of what radian measure means. His study also revealed interesting results about π. Some of the participants considered π as the unit for the radian measure and considered 1 radian equalled to 180°.

Topcu et al. [9] specifically focused on the radian concept in trigonometry and found that most of the pre-service and in-service mathematics teachers' concept images of radian were not rich enough and were dominated by their concept images of degree. This study also investigated possible sources of such concept images. First of all, the interview data of this study indicated that although the participants were successful to convert between degree and radian using the equation $D/180 = R/\pi$, none of the participants defined the radian as the arc length. Secondly, participants who have stronger concept images of radian established richer connections between unit circle and other concepts in trigonometry. On the other hand, participants who have stronger degree images used the right triangle. The aim of this study is to give a full report of the findings in Topcu et al. [9] with different participants.

2. Theoretical framework

Pre-service mathematics teachers' understandings of radian were investigated under the theoretical framework of concept image. Tall and Vinner [10] introduce the notions of concept definition and concept image and make a distinction between the two. They define concept definition as the 'form of words used to specify that concept' (p. 152). A formal concept definition is one accepted by the mathematical community at large. As Tall and Vinner [10] assert, we can use mathematical concepts without knowing the formal definitions. To explain how this occurs, they define concept image as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' (p. 152). They assert that it is built up over the years by experience, and that different stimuli at different times can activate different parts of the concept image developing them in a way, which need not be a coherent whole. Vinner [11] asserts that specific individuals create idiosyncratic images and also the same individual might react differently to a concept encountered within different situations.

Under this theoretical framework, this study tries to answer the following research questions:

- What kinds of concept images of radian do pre-service teachers have?
- What are the sources of these concept images in relation to the uses of right angle and unit circle?

3. Methodology

A multiple-case study was conducted for this study. Cohen and Manion [12] describe a case study as the observation of the characteristics of an individual unit such as a child, a class, a community or a school. It allows an intensive analysis of the phenomena. More than one case might be needed in order to observe differences in the studied phenomena [13]. Therefore, a multiple-case study was conducted to investigate a diversity of concept images of radian that pre-service teachers have.

3.1. *Participants and setting*

The participants are six pre-service mathematics teachers attending a mathematics teacher education programme in a university in Istanbul, Turkey. Candidates enter this

5 year programme on the basis of their scores in a university entrance exam in which they are required to respond to questions in mathematics, science and Turkish language. After graduating from the programme, they are entitled to a certificate for teaching mathematics in high schools. The programme has two modules: pure mathematics module (in which they take courses from the department of mathematics) and education module. During the second module, they take courses from both education and mathematics education departments, which aim to develop general and subject-specific pedagogical knowledge and they do teaching in actual classrooms. The research was conducted in the 'Teaching Methods in Mathematics I' course in which the theories of learning, pedagogical content knowledge and teaching methods in mathematics are discussed. The tutor of the course is the author of this article. During the discussion of pedagogical content knowledge, pre-service teachers are first assessed on their subject knowledge on specific mathematics concepts and they are encouraged to reflect on their own understandings of these concepts. This is followed by a discussion of pedagogical content knowledge and the micro-teaching of these specific mathematical concepts by pre-service teachers. However, this study only focuses on pre-service teachers' understanding of radian concept.

3.2. *Selection of cases and data collection instruments*

The data were collected using semi-structured interviews. Six pre-service teachers were selected among a class of 42 pre-service teachers on the basis of a questionnaire they have completed. Five of them are female and one of them is male. Their ages ranged from 22 to 23. The names of the pre-service teachers were changed to Aysel, Eren, Canan, Sena, Selcan and Belma. The questionnaire aimed to assess their concept images of radian (Appendix). Questions were selected from Topcu et al. [9] except the questions 7, 8, 9 and 10, which were added for this study.

On the basis of the responses to the questionnaire, six pre-service teachers were selected for individual interviews using the method of theoretical sampling. As Mason [14] asserts theoretical sampling, which is a kind of purposeful sampling, means selecting a sample on the basis of their relevance to the research questions and theoretical positions to be able to build in certain characteristics or criteria, which help to develop and test the theory and explanation. Since the first research question aims to investigate the concept images of radian, pre-service teachers who have different concept images were selected, on the basis of the analyses of the responses to the questionnaire, to be able to explore a diversity of concept images. A detailed account of the results from the questionnaire is beyond the scope of this article. However, to explain the selection procedure, the results from the analyses of questions 1, 6 and 10 will be presented briefly in this section. Forty-two pre-service teachers' responses to the questionnaire were analysed to describe their concept images of radian. Among the 42 pre-service teachers, only 1 pre-service teacher (Aysel) could explain radian as the length of an arc as a response to question 10 (Appendix). Therefore, Aysel was selected as the first case. The results from the analyses of questionnaires also showed that majority of the participants' concept images were dominated by their strong degree images. For instance, for question 1 (Appendix), 39 out of 42 pre-service teachers considered $\sin 30$ as $\sin 30^\circ$, although the angle is represented without the degree notation (Table 1).

As can be seen from the table, majority of pre-service teachers considered angles in radian only when they included π. Otherwise, they considered angles, when given as real

Table 1. Pre-service teachers' responses (frequencies) to question 1 in the questionnaire ($N = 42$).

	Considering the angles in radian	Considering the angles in degree	Neither in degree nor radian	Other	No response
1(a) $\sin 30$	–	39	1	2	–
1(b) $\sin 60$	–	39	1	2	–
1(c) $\sin \pi/2$	38	–	–	4	–
1(d) $\sin \pi/3$	40	–	–	2	–
1(e) $\sin 3$	–	37	–	4	1

Table 2. Pre-service teachers' responses (frequencies) to question 6 in the questionnaire ($N = 42$).

Categories of responses	Frequencies
Using the unit circle to define the cosine function.	8
Using the right triangle to define the cosine function.	17
Mentioning only the domain and range without an explanation.	10
Other.	3
No response.	4

numbers such as $\sin 30$, in degrees. This indicates that most of the pre-service teachers have concept images, which were dominated by concept images of degree.

Since the second research question aims to investigate sources of these concept images in relation to the uses of right angle and unit circle, pre-service teachers were selected from those who used right triangle or unit circle, respectively. Out of 42 pre-service teachers, only 8 of them used the unit circle to define the cosine function in question 6 (Table 2). On the other hand, 17 of them used the right triangle and defined the cosine function as the ratio of the adjacent side over the hypotenuse.

Considering these results, Aysel was selected as the only pre-service teacher who has a strong concept image of radian and who uses unit circle to define cosine function, and the other five pre-service teachers were selected as those who have a strong concept image of degree measure. Among those five pre-service teachers, Belma, Canan, Sena and Selcan were selected, since they used the right triangle to define the cosine function and Eren was selected, since he used the unit circle to define the cosine function. This way, six pre-service teachers were selected to represent a diversity of concept images.

Concept images and sources of those concept images were examined qualitatively using semi-structured interviews. Six pre-service mathematics teachers were interviewed using the same questions in the questionnaire (Appendix). It was aimed to carry out on-the-spot analysis so that follow-up questions can be asked to reveal interviewees' concept images. During the interview, interviewees were asked to think aloud while they were solving the questions on the article.

3.3. *Data analysis*

The interviews lasted around an hour and were tape-recorded. Verbatim transcripts of the tapes were open-coded using line by line analysis and memos were written considering the research questions. Themes were identified considering the memos.

4. Results from the interviews

In this section, a detailed account of the data obtained from the interviews will be reported in two sections: a section for concept images of radian for six cases and another section on the sources of these concept images in relation to the use of right triangle and unit circle.

4.1. *Results from the interviews on the concept images of radian*

4.1.1. *Case of Aysel*

Aysel was selected as the most successful pre-service teacher who responded to the questionnaire. Among the 42 pre-service teachers, only she could define radian as the length of an arc: 'radian is the angle measurement which is equal to the arc length on the unit circle'.

Although she defined the radian as above and explained it by her drawing as shown in Figure 2, her responses to other questions indicated that she has concept images dominated by her degree images. In the first question, she could successfully evaluate $(\pi/2, f(\pi/2))$ and $(2, f(\pi/3))$ when $f(x) = x \sin x$, and plotted these points correctly on the Cartesian plane. However, she could not evaluate $(30, f(30))$, $(\pi/6, f(60))$ and $(3, f(3))$ when the angles were expressed in real numbers without degree notation.[1] She evaluated the expressions and plotted the points $(30, f(30))$, $(\pi/6, f(\pi/6))$, $(\pi/2, f(\pi/2))$, $(2, f(\pi/3))$, $(3, f(3))$ on the Cartesian plane as shown in Figure 3:

> Aysel: I'm not sure how to evaluate these since the angle measures such as degree or radian were not mentioned… At first, I wrote sin 30 as 1/2, and evaluated '30 sin 30' as 15. But I'm not sure, so I leave it as 30 sin 30. However, it doesn't change the order (on the y-axis) for

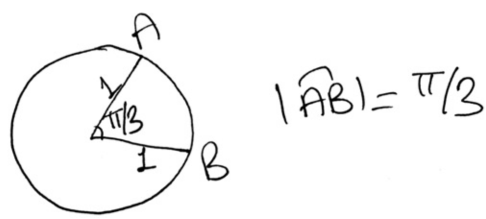

Figure 2. Aysel's drawing to explain the concept of radian.

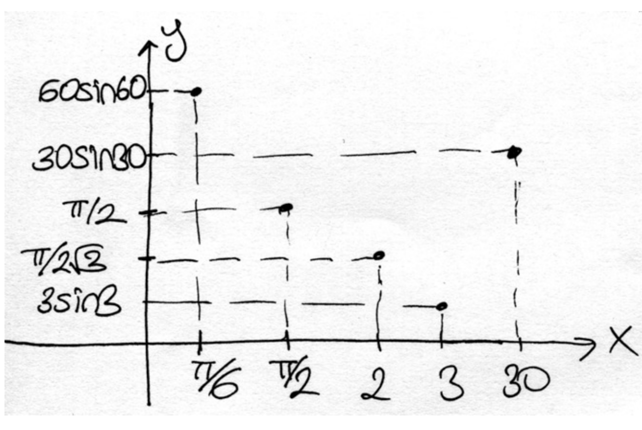

Figure 3. Aysel's plotting for question 1.

> my plotting … I leave 3 sin 3 as it is since it is given as a real number and no angle measurement(degree or radian) was mentioned.

In the interview, the absence of degree notation was emphasized. However, as can be seen from her responses above, she could not consider angles in real numbers as radian measure. On the other hand, she considered real numbers such as $\pi/2$ in radian. She said that 'It's not a problem when I evaluate $(\pi/2)(\sin \pi/2)$ since it was given in radian'.

In questions 2, when $f: R \to R$, $f(x) = \cos x$ is defined and $\cos x = -\sqrt{3}/2$ is given, she successfully expressed x as $\{5\pi/6 + 2k\pi,\ k \in Z\}$ and $\{7\pi/6 + 2k\pi,\ k \in Z\}$. Similarly, in question 3, she expressed arctan (1) and $\arctan(-\sqrt{3})$ as real numbers, not in degrees. When she was asked why she expressed the values in radian rather than degree, she said that using radian was easier than using degree measure and is more appealing. However, she could not relate it to the fact that the domain of the function $f(x) = \cos x$ are real numbers and therefore, the x values cannot be expressed in degrees.

In question 4, she found $\mathrm{Lim}_{x \to \pi} f(x)/(2x - \pi)$ as $\cos(\pi^2/2)$ when $f(x) = (x - \pi/2)$. $\mathrm{Cos}(\pi x)$. However, she could not give meaning to $\cos(\pi^2/2)$. She said that:

> Aysel: At first, I thought I couldn't find this limit since the function is defined on real numbers and we have to substitute π in $f(x)$. But π is a real number, so I can find it. If I substitute π in $f(x)$, then I find $\cos(\pi^2/2)$.
>
> Researcher: So you have found it as $\cos(\pi^2/2)$. What can you tell me about the value of $\cos(\pi^2/2)$?
>
> Aysel: I cannot say anything. I would leave it as it is. I cannot go further. I haven't thought of an expression like this before.

As seen in her responses, she could not give meaning to $\cos(\pi^2/2)$ by seeing one of the π as the coefficient of the other for $\cos(\pi \cdot \pi)$, that is π times π round the unit circle, which gives a value close to $\cos 3\pi$. Reason for her difficulty might be that π evokes distinct images: π as a value of an angle in radian and as an irrational number π ,which is close to 3.14. She could not relate these distinct images in the same context, therefore, could not treat one of the π's as the coefficient of the other π, which is an angle in radian. This claim will be investigated for the other five cases throughout the article.

In summary, it can be concluded for Aysel that although she can define the radian concept as the length of an arc, she has concept images of radian dominated by the concept images of degree.

4.1.2. *Case of Eren*

Eren could convert the given angles in radian to angles in degree measure and vice versa. However, he did not define radian as the length of an arc on a circle. He defined it as follows: 'Considering the circumference of 2π of a unit circle, a 2π part of it is 1 radian'.

In question 1, although it was mentioned that the function $f(x) = x \sin x$ is defined on real numbers, he evaluated the values in degrees for (a), (b) and (e) as presented in Table 3. He plotted the points $(30, f(30))$, $(\pi/6, f(\pi/6))$, $(\pi/2, f(\pi/2))$, $(2, f(\pi/3))$, $(3, f(3))$ as shown in Figure 4.

As seen from his drawing, he plotted $\pi/2$ as 90 on the x-axis. Therefore, later in the interview, he was asked what π evoked to him:

> Eren: π is 3.14 and $22/7 \cdot \pi = 180°$. It is the constant, and an irrational number, that we use to find the area and circumference of a circle. It is the constant used to express angles in terms of radian measure.

Table 3. Eren's responses to question 1.

(a) $f(x) = x \sin x \, f(x) = 30 \sin 30 = 30\left(\frac{1}{2}\right) = 15$
(b) $f(60) = 60 \sin 60 = 60 \cdot \frac{\sqrt{3}}{2} = 30\sqrt{3}$
(c) $f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) \; f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)$
(d) $f\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \pi\sqrt{3}/6$
(e) $f(3) = 3 \sin 3 \, f(3) = 3 \sin 3$

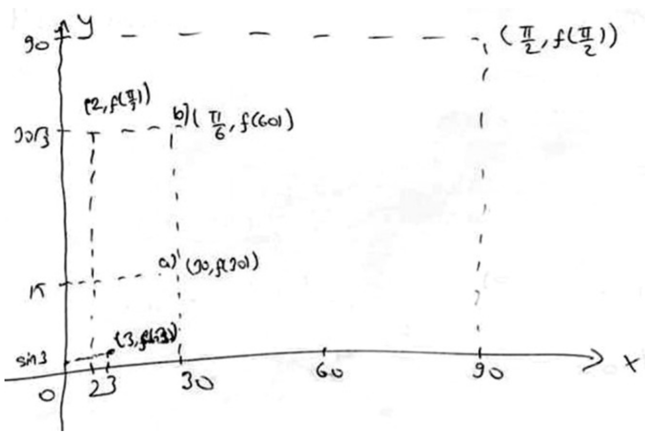

Figure 4. Eren's plotting for question 1.

Researcher: How is π as an irrational number and π as 180 degrees related?
Eren: π is equivalent to 180 degrees.
Researcher: So it's not equal to 180 degrees.
Eren: The circumference of the unit circle is 2π. One complete turn is also 2π and this is equivalent to 360 degrees so π is equivalent to 180 degrees.

As seen from his responses, he could not clearly relate his two different images of $\pi : \pi$ as an irrational number and π as an angle. Although he sees π as equivalent to 180°, he could not consider the irrational number π as the arc length of π on a unit circle.

In question 2, when $\cos x = -\sqrt{3}/2$ is given, he evaluated the value of x both in degree and radian. Similarly, in question 3, he found arctan (1) and arctan$(-\sqrt{3})$ both in degrees and radian. In question 4, when $f(x) = (x - \pi/2) \cdot \cos(\pi x)$ is given, he found $\lim_{x \to \pi} f(x)/(2x - \pi)$ as $\cos(\pi^2/2) = 1/2$. When he was asked how he found it, he stated that '$\cos \pi$ equals to -1, -1 squared is 1'. He was reminded that he should evaluate $\cos \pi^2$, not $\cos^2 \pi$. However, he said: 'in that case, I would leave it as it is, because I don't know what to do with $\cos \pi^2$'.

In summary, Eren could not define radian concept as the length of an arc on a circle and has a strong concept image of degree. As a result, he has two distinct concept images of π, π as a number and π as an angle, which he could not relate to each other.

Table 4. Canan's responses to question 1.

(a) $f(x) = x \sin x$ $f(x) = 30 \sin 30 = 30\left(\frac{1}{2}\right) = 15$
(b) $f(60) = 60 \sin 60 = 60\left(\frac{\sqrt{3}}{2}\right) = 30\sqrt{3}$
(c) $f(x) = \left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$ $f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)$
(d) $f\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \pi/2\sqrt{3}$
(e) $f(3) = 3 \sin 3$ $f(3) = 3 \sin 3$

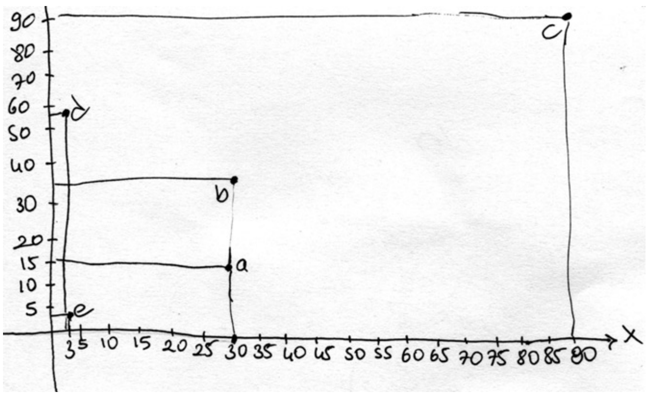

Figure 5. Canan's plotting for question 1.

4.1.3. *Case of Canan*

Canan defined radian as a kind of angle measurement. She said that she could not remember the definition but she could relate it to a circle, which has a semi-circumference with the length of π. She added that the radian concept evokes to her 180° which is π: 'in the unit circle, the radius is 1 and the circumference is 2π which gives us one complete turn'. It is obvious from her response that she related the radian concept to unit circle. However, she could not explain radian concept as the length of an arc but rather a type of angle measurement, which she explains in terms of degree. This indicates that her concept image of radian is dominated by her degree image.

In question 1, although it was mentioned that the function $f(x) = x \sin x$ is defined on real numbers, she evaluated the values in degrees for (a), (b) and (e) as in Table 4. As seen from her responses, she evaluated the angles in radian only when they were expressed in terms of π. Otherwise, she considered angles in degrees. She was reminded once again that there was no degree notation after 3, 30 and 60, but she still considered sin 3, sin 30 and sin 60 in degrees. This is obviously because of a lack of understanding of radian concept and a strong concept image of degree. She plotted the points she found as shown in Figure 5.

As seen from her drawing, she plotted $\pi/2$ as 90 on the x-axis. Later in the interview, a discussion on π has occurred for question 8 as follows:

Canan: π evokes 180 to me.
Researcher: What is the number π?
Canan: It's close to 3.14..., goes on like this

Researcher:	So it's both 3.14... and 180?
Canan:	180 degrees, it's equivalent to π radian
Researcher:	You mentioned two different values. Are there two different π's?
Canan:	No they're not. They're the same.
Researcher:	Could you explain to me why they are the same? π as a number and π as an angle in radian?
Canan:	I don't know, but they are the same
Research:	In question 2, you expressed x as $150+2k\pi$. For instance, if k is 1 then you get $150+2\pi$. So what do you get here?
Canan:	2π is 360 degrees, so it's $150+360$. After 150, there is one more turn around.

As seen from her responses, she distinguished between π in the context of angle and π in the context of number. Although she mentioned that it was the same π, she could not explain how they are related. This confusion became quite clear in her response to question 4 in which she was asked to comment on the value of $\cos(\pi^2/2)$. She tried to figure out whether one of the π's (as the coefficient of the other π) is odd or even. She then considered it as 180 and evaluated $\cos \pi^2$ as 1. This indicates that she considered π not only as equivalent to 180° but also as equivalent to the number 180. The way she used π and 180 interchangeably is an indication of her strong degree image. In summary, although Canan could convert between degree and radian angles, she could not deal with angles when given as real numbers without degree notation and has a strong concept image of degree.

4.1.4. *Case of Sena*

Sena could successfully convert the given angles in radian to angles in degree measure and vice versa. However, she could not define the radian concept as the length of an arc on a circle. Instead, she defined the radian concept as a type of an angle measure. In question 1, she was asked to evaluate $f(3)$, $f(30)$, $f(60)$, $f(\pi/2)$ and $f(\pi/3)$ when $f(x)=x \sin x$ is given. She evaluated these values as presented in Table 5.

As can be seen in Table 5, she successfully evaluated the expressions that included π. On the other hand, she could not find the values of $f(3)$ $f(30)$, $f(60)$ and this is exemplified with the excerpts below:

Sena:	If we substitute 30 here, it's just a number. It is not in degrees or in radian. We don't know what it is. It's not clear. Therefore I leave it as it is. $f(\pi/2)=(\pi/2)\sin(\pi/2)$.
Researcher:	You have left $f(30)=30\sin 30$ as it is but you found the values of sine when the angles are expressed in terms of π.

Table 5. Sena's responses to question 1.

(a) $f(x)=x\sin x$ $f(x)=30\sin 30$
(b) $f(60)=60\sin 60$
(c) $f(x)=\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$ $f\left(\frac{\pi}{2}\right)=\left(\frac{\pi}{2}\right)$
(d) $f\left(\frac{\pi}{3}\right)=\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)=\left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right)=\pi/2\sqrt{3}$
(e) $f(3)=3\sin 3$ $f(3)=3\sin 3$

Sena: We can find $f(\pi/2)$ because a value such as $\pi/2$ is given. I imagined the unit circle. We know that it is 90 in degrees. It's convertible to degree measure but $f(3) = 30\sin 30$ is not clear.

As seen from her responses, she could find the values of sine function when the angles are expressed in terms of π. She could do that by converting the angles to degree measure e.g. considering $\sin(\pi/2)$ a $\sin 90^\circ$. However, she could not find the values of sine function when the angles are expressed as real numbers. It is obvious from her responses that she has a strong concept image of degree. Therefore, she could not deal with the angles given as real numbers unless they included π.

When she plotted the points on the Cartesian plane, she plotted $\pi/2$ as 90 on the x-axis (Figure 6). Her way of thinking of π as 180 is obviously an indication of her strong concept image of degree. During the interview, she was asked what π evoked to her. She said the following:

Sena: 3.14 or 22/7 and 180 come to my mind. We use π to find the area and circumference of a circle. It's an important constant that we use in math.
Researcher: You mentioned two different numbers for π. 3.14 and 180. How come they represent the same number π.
Sena: As a number it's 3.14 and in geometry it's 180 degrees.
Researcher: Can you explain how they are related?
Sena: erm . . . don't know.

As obvious from her responses, she has two different concept images of π, which she could not relate to one another: one in the context of angle and one in the context of number. In the context of angle, she associates π to 180° and this indicates her strong concept image of degree over radian. Her responses to questions 2 and 3 have also revealed this concept image. In question 2, she evaluated the value of x in degrees rather than radian when $\cos x = -\sqrt{3}/2$ is given. Similarly she found arctan(1) and arctan($-\sqrt{3}$) in degrees as 45° and 150°.

In question 4, she found $\lim_{x\to\pi} f(x)/(2x-\pi)$ as $\cos(\pi^2/2)$ when $f(x) = (x - \pi/2).\cos(\pi x)$. However, she could not evaluate the value of $\cos(\pi^2/2)$:

Researcher: What is $\cos\pi^2$?
S: Well. Actually we take π as 180 but it's not clearly given here. It's gonna be complicated when we evaluate it with a calculator as

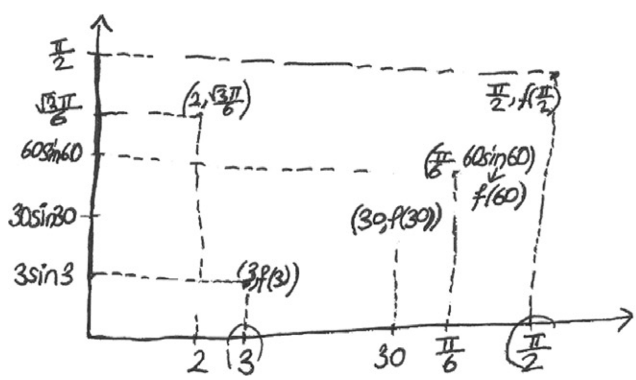

Figure 6. Sena's plotting for question 1.

Table 6. Selcan's responses to question 1.

(a) $f(x) = x \sin x$ $f(x) = 30 \sin 30 = 30\left(\frac{1}{2}\right) = 15$
(b) $f(60) = 60 \sin 60 = 60\left(\frac{\sqrt{3}}{2}\right) = 30\sqrt{3}$
(c) $f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$ $f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)$
(d) $f\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = (\pi\sqrt{3}/6)$
(e) $f(3) = 3 \sin 3$ $f(3) = 3 \sin 3$

> $\cos 180.180$. It might be 3.14 times 3.14. It's not clear. I don't know. $\cos(\pi^2/2)$ doesn't make sense to me.

As seen from her responses, Sena could not distinguish between π in the context of angle and π in the context of number, therefore, she could not find the value of $\cos \pi^2$ by treating one of the π's as the coefficient (as a real number) and the other π as the angle. In summary, Sena did not know how to deal with sine and cosine of real numbers unless they are expressed in terms of π and she was confused with two different values of π: 180 and 3.14. Therefore, for Sena, it can be concluded that she has a strong concept image of degree.

4.1.5. *Case of Selcan*

Selcan could successfully convert the given angles in radian to angles in degree measure and vice versa. Although she explained radian on a unit circle by defining 2π radian as an angle of one complete turn and π as the ratio of the length of circumference of a circle to its diameter, she could not define the radian as the length of an arc on a circle.

In question 1, although it was mentioned that the function $f(x) = x \sin x$ is defined on real numbers, she evaluated the values in degrees for (a), (b) and (e) as in Table 6. She then plotted the points a, b, c, d, e for $(30, f(30))$, $(\pi/6, f(\pi/6))$, $(\pi/2, f(\pi/2))$, $(2, f(\pi/3))$, $(3, f(3))$ as shown in Figure 7.

She reflected on her drawing as follows:

> Selcan: What is confusing is that the function is defined from real numbers to real numbers. It should be $\sin 30^\circ$ but it's just 30. That's why I was unsure. To be an angle, it should be either degree or radian. But here, it's none of them. Just $\sin 30\ldots$ when plotting the points I considered $3 \sin 3$ close to 0 since the sine function is increasing as you go from 0 to 90 degrees.

As can be seen in her responses above, she considers angles in radian only when they are expressed in terms of π. Otherwise she considered angles in degrees. In question 2, when $\cos x = -\sqrt{3}/2$ is given, she evaluated the value of x both in degrees and radian. Similarly, in question 3, she found $\arctan(1)$ and $\arctan(-\sqrt{3})$ both in degrees and radian. Although she was reminded that the values of arctangent should be real numbers, she still considered $45^\circ + k360^\circ$ as an answer.

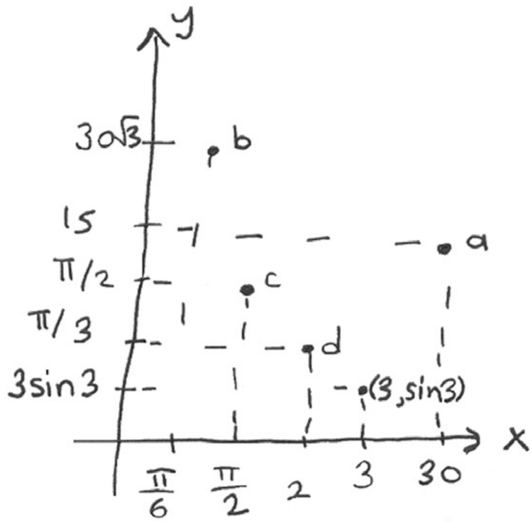

Figure 7. Selcan's plotting for question 1.

In question 4, she found $\lim_{x\to\pi} f(x)/(2x-\pi)$ as $\cos(\pi^2/2)$ when $f(x)=(x-\pi/2).\cos(\pi x)$. She left $\cos(\pi^2/2)$ as it is. She said that she could find the value of $\cos\pi$ but could not decide what $\cos\pi^2$ is. During the interview she was asked what π evoked to her and she stated the following:

Selcan: First thing that comes to my mind is the way we find the circumference of a circle. It's 2π times the radius. As long as I know, π is the ratio of the length of the circumference over the diameter of a circle, I mean any circle. π is a constant and it doesn't change for different circles. This constant is 22/7 which is approximately 3.14... it is also equivalent to an angle of 180° in trigonometry.

Researcher: So how these two, π as 3.14 (approximately) and 180°, are related?

Selcan: I don't know exactly. I never thought about this before.

As can be inferred from her responses, Selcan has two different concept image of π, which she does not relate to each other: π as $\approx$3.14 (incorrectly considered π as 22/7) and π as an angle equivalent to 180°. In summary, it can be concluded that Selcan has strong concept image of degree.

4.1.6. *Case of Belma*

Belma could successfully convert the given angles in radian to angles in degree measure and vice versa. However, she could not define the radian concept as the length of an arc on a circle. Instead, she defined radian as a type of an angle measure. She said that angles in radian are expressed in terms of π.

In question 1, although it was mentioned that the function $f(x)=x\sin x$ is defined on real numbers, she evaluated the values in degrees for (a), (b) and (c) as in Table 7. She plotted the points $(30, f(30))$, $(\pi/6, f(\pi/6))$, $(\pi/2, f(\pi/2))$, $(2, f(\pi/3))$, $(3, f(3))$ as shown in Figure 8.

Table 7. Belma's responses to question 1.

(a) $f(x) = 30\sin 30 = 30\left(\frac{1}{2}\right) = 15$
(b) $f(60) = 60\sin 60 = 60\left(\frac{\sqrt{3}}{2}\right) = 30\sqrt{3}$
(c) $f(x) = \left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\ f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)$
(d) $f\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = (\pi/2\sqrt{3})$
(e) $f(3) = 3\sin 3\ f(3) = 3\sin 3,\ (3, 3\sin 3)$

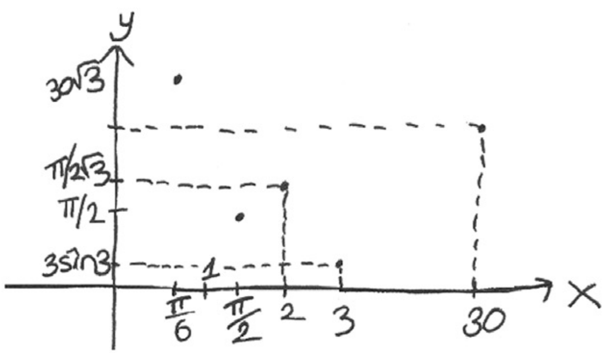

Figure 8. Belma's plotting for question 1.

She mentioned that she assumed that the angles are in degree measure. Therefore, she was reminded that the function $f(x) = x\sin x$ is defined on real numbers and she was asked not to consider them in degree measure but as real numbers:

Researcher: There's no degree notation there, so assume that they are numbers without degree notation. What can you say about $f(30) = 30\sin 30$. It's not $30°$, just $\sin 30$.

Belma: Let me say, in trigonometry we always assumed that x in sine function is in degrees... we automatically see it in degrees, we were not told otherwise. If we consider these values as real numbers, then we would not get the outputs of sine function, I think.

Researcher: The sine function is defined from R to R. But you say we can not get the sine values if we take 30 as a real number and should consider $30°$ instead.

Belma: It was told us as a rule, when we see 30 we should consider $30°$. $f(30)$ means $f(30°)$. Because if a function is defined from real numbers to real numbers, then sin(30) should give a real number. If it's not $f(30°)$ then we cannot get a real number.

Researcher: Well, let's consider $\sin(\pi/2)$. Here $\pi/2$ is a real number and you evaluated it as 1 in question 1c. So it gives us a real number.

Belma: That's something I didn't understand in high school. It's the number π. π as 3.14 and π as radian seems different to me. π is equivalent

> to 180 degrees... so in expressions which includes sine or cosine I see π as 180 degrees. When you say $\sin(\pi/2)$, $\sin 90$ comes to my mind. I never think it as $\sin(1.5)$ or something closer to that. That's how I was taught. That's the way I think. I always think it as $\sin 90$.

As seen from her responses, Belma has two different images of π. The first image of π is the real number π , which she could plot on the x-axis near 3.14 (Figure 8). The second image of π is related to an angle equivalent to 180°. As a result of these images of π, she sees $\sin(\pi/2)$ not approximately as $\sin 3.14$ but as $\sin 90°$.

In question 2, when f: $R \rightarrow R$, $f(x) = \cos x$ is defined and $\cos x = -\sqrt{3}/2$ is given, she first expressed x in degrees as 150° using a unit circle. She then expressed it in radian as $5\pi/6$, since she wanted to express it as a real number. Similarly, in question 3, she first expressed it in degrees and then converted it to radian measure. She mentioned that she converted the degree measures to radian since the functions are defined on real numbers. This awareness about radian concept and its relation to real numbers was not apparent in the beginning of the interview and might have been caused by the dialogue above for question 1.

In question 4, she found $\lim_{x\rightarrow\pi} f(x)/(2x-\pi)$ as $\cos(\pi^2/2)$ when $f(x) = (x - \pi/2)$. She first evaluated $\cos \pi^2$ as -1. She mentioned that:

Belma: For cosine, any number times π gives -1. π is a real number, so cosine of 3.14 times π makes -1. If it's an even number, it is 0° or 360°. It's periodic, it is either 180° or 360°.

As seen from her response, Belma treated one of the π's as the coefficient (as a real number) and the other π as the angle. However, she could not decide how many times it should turn around the unit circle. For the case of Belma, it can be concluded that she could not define the radian concept as the length of an arc and she has a stronger concept image of degree.

4.2. *An overview of the concept images for the six cases*

In the sections above, an analysis of the data about the concept images of radian for six cases was demonstrated. As the first case, Aysel could define the radian concept as the length of an arc on a circle. However, her concept image of radian is still dominated by her concept image of degree. For the other five cases, Eren, Canan, Sena, Selcan and Belma were successful with converting between radians and degrees. However, instead of explaining radian as an arc length on a circle, they saw radian as a type of angle measurement, which can be explained in terms of degree.

The themes that emerged from the analyses of the interview data for the six cases indicate that the pre-service teachers' concept images of degree dominate their concept images of radian. These themes are summarized below:

- Always considering angles (without the degree notation) in degrees except there is π in the expression even though the absence of degree notation is emphasized. As a result, one can consider π as a unit of radian.
- Plotting π on the x-axis as 180 instead of plotting π near 3.14.
- An understanding of an angle in radian by relating it to its equivalent in degree.
- Having two different concept images of π in two different contexts: one in the context of angle and one in the context of number. In other words, distinguishing π as a value of an angle and π as a real number.

4.3. *An investigation of the sources of concept images*

Topcu et al. [9] suggest that sources of concept images of degree and radian are closely related to the use of right triangle and unit circle. They found that participants who have stronger concept images of radian established richer connections between unit circle and other concepts in trigonometry while participants who have stronger degree images have right triangle as one of their cognitive units. Therefore, in this study, to investigate the sources of participants' concept images, their understanding and use of right triangle and unit circle were examined considering their responses to questions 6, 9 and 10 during the interview (Appendix).

4.4. *Results from the interviews on the use of right triangle and unit circle*

In this section, the findings on the use of unit circle and right triangle will be reported for the six cases. During the interview, participants were asked the place and importance of unit circle and right triangle for the teaching of trigonometry (see questions 9 and 10 in Appendix). They were also asked to define the cosine function to examine how they relate the unit circle and/or right triangle to trigonometric functions (see question 6 in Appendix).

In the interview, Aysel said that she used the right triangle only for finding the specific values of sine and cosine. However, for the unit circle, she stated that she relates the unit circle to various concepts in trigonometry such as finding the values of sine, cosine, tangent and cotangent functions, determining the domain and range of these functions and algebraic relationships among trigonometric functions such as $\sin^2 x + \cos^2 x = 1$:

> Aysel: We can define the trigonometric functions and explore their properties by using the unit circle. For instance, the image of the functions $f(x) = \sin x$ and $f(x) = \cos x$ are $[-1, 1]$. We can find this from the unit circle. Similarly, we can find that the function $f(x) = \tan x$ is undefined at $\pi/2$. These can all be explained to students by using the unit circle. Also, algebraic relationships among trigonometric functions can also be clearly seen using the unit circle. For instance, we can construct a right triangle inside the unit circle and can prove that $\sin^2 x + \cos^2 x = 1$ by using the Pythagoras theorem. Apart from these, we can find some values of trigonometric functions such as sine, cosine, tangent, cotangent. Visual explanations of these are more appropriate. For instance, we can find the value of $\sin 120^\circ$ using the unit circle.

As seen in her responses, she relates unit circle to various concepts in trigonometry. When defining the cosine function in question 6, she used the unit circle to define the domain and range of the function. She also drew the graph of cosine function using the unit circle:

> Aysel: If we think of the unit circle, when I take an angle, to find its cosine means to find the x-coordinate of the point where the side of angle intersects the unit circle. Since these points are on the unit circle, the x-coordinate of this point can range from -1 to 1.
>
> Researcher: So you define cosine function from real numbers to real numbers. What do you find for a real input? For example, in question 1, you said you would leave $30 \sin 30$ as it is, since 30 is a real number but not given in degree. When you were evaluating this, did you consider that the sine function is defined from real numbers to real numbers?

Aysel: No, I didn't. I first considered sin 30 as 1/2. Then I thought that it wasn't given in degrees therefore left it as it was.

Researcher: Let's think about it again and remember that you defined cosine function on real numbers. Like sin 30. Is 30° a real number?

Aysel: Erm ... let me think. It's a real number but with the degree notation. But we just think it as a unit of angle measure.

Researcher: Consider it without the degree notation, just as a real number.

Aysel: erm ... let me think of it on the graph of cosine. For π it takes a value of -1. Can I find an approximate value from here? π is approximately 3.14. Consider 2π, 3π, it repeats itself as you go around the unit circle. You would get 30, after approximately 10 turns, 31.4. Therefore, cos 30 will be around 1. It's gonna be bigger than 1.

As seen from her responses although Aysel defined the radian as the length of an arc as discussed in the previous sections, she had difficulties in considering 30 as an angle in radian. In the interview, after she defined the cosine function from real numbers to real numbers using the unit circle, she was encouraged to re-think about the value of sin 30. During the interview, focusing on the graph of cosine function and the unit circle, she came to realize that cos 30 can be considered without the degree notation and cos 30 is $\approx$1. As we will discuss below, the other pre-service teachers could not evaluate the value of sine or cosine of angles given as real numbers.

Eren related various concepts in trigonometry both to unit circle and right triangle:

Eren: Right triangle is useful to find various trigonometric values. First consider the triangle of 30–60–90 and 45–45–90. They are helpful to students to recall basic trigonometric values such as sin 60 degree, cos 45 degree. This way, students do not need to memorise these. Also, they can try to discover formulas such as $\sin^2 a + \cos^2 a = 1$ and $\tan x \cdot \cot anx = 1$ ... Working with a unit circle makes our job easier. For instance, the image of sine and cosine function is $[-1, 1]$ and this can be explained on the unit circle. This way, students can understand trigonometric functions better. Similarly, tangent and cotangent is represented by the lines $x = 1$ and $y = 1$. We can also find specific values such as sin 30, cos 60, tan 45, cos 0, sin 90 from the right triangle inside a unit circle.

As can be inferred from his excerpts, Eren emphasized the importance of both unit circle and right triangle and used both of them to relate concepts in trigonometry. He preferred to use the unit circle to determine the domain and range of trigonometric functions while he used the right triangle to find the specific values of trigonometric functions such as cos 60°. In question 6, he defined cosine function as defined from $R \rightarrow [-1, 1]$. He stated that the cosine function is represented by the x-coordinate on a unit circle, therefore takes values between -1 and 1. However, as he mentioned, he reached this conclusion by constructing a right triangle inside a unit circle and this indicates his preference for using right triangle.

In case of Canan, although she mentioned that she uses the unit circle to find the values of trigonometric functions, she used it to construct a right triangle with a hypotenuse of length 1 inside it:

Canan: When we started to learn trigonometry in high school, after we've learnt the right triangle, we used to draw unit circle on the Cartesian plane to find the sine and cosine of angles. But I didn't understand how this was done . . . I only use the unit circle to construct a right triangle inside it and show that the length of the hypotenuse is 1.

Researcher: Is there any other issues you want to mention about the importance of unit circle for the teaching of trigonometry.

Canan: For instance, if we think of the circumference of the circle, it is 2π since the radius of the unit circle is 1. That is why it's important.

Researcher: How do you relate the radian concept to the circumference of the unit circle which is 2π?

Canan: Radian is expressed in π or 2π and the circumference of a unit circle is 2π.

Researcher: Can you give the definition of radian?

Canan: No, I don't know

As seen above she said she used the unit circle for only constructing a right triangle inside it and did not know how to evaluate sine and cosine values using the unit circle. Likewise, in question 6, she defined the cosine function using the right triangle as the ratio of the length of adjacent side over hypotenuse.

Sena considers the unit circle as the starting point of trigonometry:

Sena: It's a starting point of trigonometry. We find the values of trigonometric functions using the unit circle. It helps students to learn and recall. When working on a question, using the unit circle helps reviewing the information you have. It's a kind of visualization of trigonometry. For instance, we can discover where the trigonometric functions are negative and positive.

Although Sena sees using the unit circle as a starting point in trigonometry as she mentioned above, she related the definition of basic trigonometric functions such as sine and cosine to the right triangle. In question 6, she used the right triangle to define the cosine function:

Sena: The right triangle is useful to define the basic trigonometric functions. It provides practicality in trigonometry . . . when x is the angle, $\cos x$ is the ratio of the length of the adjacent side over the length of the hypotenuse of a right triangle.

Researcher: What are the domain and range of this function?

Sena: To decide where it is defined, let's decide where it's undefined. Can we say from π to $-\pi$?

Researcher: How did you decide?

Sena: I'm thinking of the unit circle. It's defined at 0 but it's not clear where it's undefined. It needs a little examination. Which values does it take and where? We need to draw its graph to see, for instance, at which value the function hasn't a value. There must be points where it's undefined.

Research: How do you decide?

Sena: From the right triangle. To be undefined, the denominator that is the length of the hypotenuse must be 0. But then, there can't be such a right triangle. Hypotenuse can't be 0. Therefore, it's not undefined at 0. But if we think of in terms of degree using the unit circle, we can find something.

Researcher: When using the right triangle, don't you think in terms of degree?
Sena: No, I consider the length there, that is adjacent side over hypotenuse, but for the unit circle I consider the angles in degrees.
Researcher: OK then, let's use the unit circle. What's the meaning of cosine of an angle on a unit circle?
Sena: Well, we do the same thing, adjacent side over hypotenuse of an angle. We draw a right triangle inside the unit circle, the radius becomes the hypotenuse.
Researcher: So at which points is the cosine undefined.
Sena: Don't know.

As seen from above, Sena could not explain the definition of cosine function on the unit circle and she relied on the right triangle to explain how the cosine function is defined.

Selcan explained the importance of unit circle as follows:

Selcan: Unit circle is important for finding the values of trigonometric functions such as sine, cosine, tangent and cotangent. By using the unit circle, we can find not only acute angles but also obtuse angles. For instance, students can find $\sin 120^\circ$ using the unit circle. However, if they use only the right triangle, they can't find it. They might need to use trigonometric formulas. Students do not need to memorise $\sin x = \sin(\pi - x)$. They can visualize it using the unit circle.

Although Selcan relates the unit circle to various concepts in trigonometry, she also considers the right triangle as the basis of trigonometry:

Selcan: It's the basis of trigonometry. We find the trigonometric ratios from the right triangle inside the unit circle. We can calculate acute angles as trigonometric ratios. When we see specific angles such as 30, 45, 60 in a triangle, we draw a line to construct a right triangle there and can find the trigonometric ratios for these angles.

In question 6, she defined the cosine function using both the unit circle and right triangle. However, as she mentioned above, she finds the trigonometric values by constructing a right triangle inside the unit circle. In summary, it can be concluded that although Selcan attaches importance to both unit circle and right triangle, she privileges the right triangle.

Although Belma emphasized the importance of unit circle for trigonometry, she mostly depended on the right triangle to define basic trigonometric functions. She mentioned that the right triangle is a tool for teaching trigonometry.

Belma: Instead of memorising, students could calculate basic values of trigonometric functions using the right triangle... although it's a tool for teaching trigonometry, we can't find most of the values of trigonometric functions. We can only make use of the right triangle for finding specific values. Another thing is that we can only find trigonometric values of acute angles. Students can't discover how to find obtuse angles from the acute angles using the right triangle. You need to know the periodicity to find $\sin(7\pi/2)$, for instance.

Although Belma related the unit circle to various concepts in trigonometry such as finding the values and signs of sine, cosine, tangent and co-tangent, she mentioned that only specific values can be found using it. As she stated below, she considers unit circle as fundamental to trigonometry and as a tool for construction of trigonometric relationships:

> Belma: Unit circle is useful to decide on the signs of the cosine and sine functions. Besides their signs, we can find the values of trigonometric functions not only for sine and cosine, but also for tangent, cotangent, secant and cosecant. Therefore it has a priority for me. It's very important in terms of practicality. It's visual. Unit circle is indispensable in trigonometry . . . if we can't remember some points in trigonometry, we can recall them from the unit circle. I don't prefer memorising and I don't want my students to memorise things.

Although she emphasized the importance of unit circle as a tool for teaching trigonometry, to define the cosine function in question 6, she first used the right triangle to explain the cosine of an angle as the ratio of the adjacent side to the hypotenuse. Although she drew the unit circle afterwards, she only used it to find the cosine values of specific angles such as 0, $\pi/2$, π and $3\pi/2$.

As can be concluded from the findings as reported above, pre-service teachers who have a deeper understanding of radian can relate and use the unit circle to various concepts in trigonometry as observed for the case of Aysel. On the other hand, pre-service teachers who have a strong degree image (Eren, Canan, Sena, Selcan and Belma) used the right triangle to explain concepts and relationships in trigonometry.

5. Discussion and implications

In this study, pre-service mathematics teachers' concept images of radian were investigated. The main finding is that the concept images of degree dominated the concept images of radian even for the case of Aysel who could define the radian concept. Having such concept images might cause problems for understanding trigonometric functions, which are defined on real numbers. As the data in this study suggested, pre-service mathematics teachers were reluctant to accept trigonometric functions with the inputs of real numbers but rather they use values in degree. An interesting finding on the strong degree image is related to π. Pre-service mathematics teachers seem to have two different concept images of π in two different contexts: π as a value of an angle and π as a real number. This kind of images of π resulted in plotting π as the number 180 on the x-axis. Such a concept image is obviously because of the difficulty in understanding of radian as a real number. In that sense, wrapping the number line onto the unit circle might be a useful metaphor to introduce trigonometric functions.

Findings of this study on the strong concept images of degree have some implications for the teaching and learning of trigonometry. Geometric underpinnings of trigonometry play an important role for the strong degree images. Students first gain experiences of angles in the context of geometry where only the degree measure is introduced. Basic trigonometric values such as sine, cosine, tangent and cotangent are first defined on the right triangle but not in the context of trigonometric functions using the unit circle. Once the unit circle and the definition of radian are introduced, trigonometric functions can be precisely defined. The findings of this study indicated that the one case, Aysel, who has a deeper understanding of radian used the unit circle to relate concepts in trigonometry. On the other hand, the other five cases, Belma, Canan, Sena, Selcan and Eren used the right triangle rather than the unit circle to define the cosine function and to relate concepts in trigonometry.

Another implication is related to teacher education. This study revealed six pre-service mathematics teachers' difficulties with radian concept. A lack of subject knowledge of radian will certainly affect their pedagogical content knowledge for trigonometry, which will affect their students' learning when they begin the profession. Therefore, as teacher educators it is important to identify these difficulties at an earlier stage and help them enrich their subject knowledge of mathematical concepts by revising the school mathematics during the teacher education programmes as parallel to their development of pedagogical content knowledge.

There is a need for future studies to explore concept images described in this study throughout the development of topics in trigonometry. For that purpose, longitudinal studies using qualitative research methods might be conducted. In such studies, teaching activities for trigonometry should be designed to eliminate concept images that limit an understanding of radian.

Acknowledgements

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Note

1. Note that some of the points are not given as $(x, f(x))$, therefore the ordinate does not depend on the abscissa for these points.

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Appendix: Questions used in questionnaire and interviews

(i) $f: R \to R$, $f(x) = x \sin x$ is given. Plot all the following points on the same Cartesian plane (points in (b) and (d) are not given as $(x, f(x))$, therefore the ordinate does not depend on the abscissa for these points). (a) $(30, f(30))$ (b) $(\pi/6, f(60))$ (c) $(\pi/2, f(\pi/2))$ (d) $(2, f(\pi/3))$ (e) $(3, f(3))$
(ii) $f: R \to R$, $f(x) = \cos x$ is given. If $f(x) = -\sqrt{3}/2$ $x = ?$
(iii) $f: R \to R$, $f(x) = \tan x$ is given. If $\tan x = a \Leftrightarrow \arctan(a) = x$ then find the following: (a) $\arctan(1) = ?$ (b) $\arctan(-\sqrt{3} =)$?
(iv) $f: R \to R$, $f(x) = (x - \pi/2) \cdot \cos(\pi x)$ is given. $\lim_{x \to \pi} f(x)/(2\pi - \pi) = ?$
(v) Convert the given angles in radian measure to degree and vice versa: $5\pi/2$, $3\pi/5$, $160°$, $50°$
(vi) Define the cosine function.
(vii) (a) Explain the concept of radian? (b) Explain the relationship between radian and degree angle measures? (Write different responses for a and (b?)
(viii) What comes into your mind when you see π? Tell me everything you know about π?
(ix) What is the place and importance of right triangle in the teaching of trigonometry?
(x) (a) What is unit circle? Explain. (b) In your opinion, what is the place and importance of unit circle in the teaching of trigonometry?