Answers of the Midterm Exam Math 255, by Dr. M. Sakalli,
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Differential Equations. Duration given was: 1hr 45 minutes. Good luck.
At the last 10-14 min a group of students were allowed to couple.

Note 1: Any question or objection welcomes.

1. Prove that the difference of any two particular solutions to inhomogeneous DE is also a solution to its homogeneous version.

$D(y_{p1}) = f(t)$, $D(y_{p2}) = f(t)$
$D(y_{p1}-y_{p2}) = D(y_{p1}) - D(y_{p2}) = f(t)-f(t)=0.$

2. Reduction of order: You have a second order homogeneous DE,

   $y'' + p(x)\, y' + q(x)\, y = 0,$

   and suppose you are given the first (nontrivial) solution as, $y_1(x)$.

a) Show that the second solution (non-proportional to $y_1(x)$) can be related by a variable $v(x)$ to the first solution which will end up with an equation of reduced order and yielding the solution of $y_2(x)$ as a function of y_1.

Answer:

$y'' + p(x)\, y' + q(x)\, y = 0$, $y_1(x)$ is proposed

And suppose $y_2 = v(x)\, y_1(x)$.

Then insert
a) y2, b) y2' = v1' y1+v1 y1', c) y2''= v1''y1 + 2y1'v1' + v1 y1'', into the equation given above. Then
= v1'' y1 + 2y1' v1' + v1 y1'' + v1' y1 p + v1 y1' p + q v1 y1
= v1'' y1 + 2y1' v1' + v1' y1 p + v1 y1'' + v1 y1' p + q v1 y1
= v1'' y1 + (2y1' + y1 p) v1' + (y1'' + y1' p + q y1) v1
= v1'' y1 + (2y1' + y1 p) v1' +   0   v1 //From the first equation
= v1'' y1 + (2y1' + y1 p) v1' = 0

This is linear and separable.

Substitute v1' = u; dv1'/dx = du/dx, which reduces the second order equation into the first order of
u' y1 + (2y1' + y1 p) u =0

The form of the equation is
A u' + B u =0; du/u = -(B/A)dx
Finding the general equation, u = e^{(-B/A)x} + c ➔ v(x) = int (u(x)) dx

Another way to figure out the equation u' y1 + (2y1' + y1 p) u =0 is Bernoulli
u' + P(x) u =0 ➔ u(x) = u(0) e^{int P(x) dx}

Then substitute ➔ v(x) = int (u(x)) dx

u = e^(-B/A) + c = C, C is another constant.
is u' y1 =- (2y1' + y1 p) u
u'/u = -2y1'/y1 + p
lnu = - 2lny1 + int(p)dx
y1^2 u = e^p.

b) This is an application of the question given above. DE you are given is

x^2 y'' -3 x y' + 4y = 0, and the first solution is $y_1 = x^2$.

i) Show that if $y_1 = x^2$ is a solution.
Answer:
This indicates that x^2 *2 -3 x (2 x) + 4 x^2 = 0; y_1 is a solution.

ii) If it is, then find y_2 by reducing the order. If it is not suggest a solution. And then proceed.
Answer:
Then y2= v y1 = vx^2; y2' = v'y1+v y1' = v' x^2 + 2 vx;
y2'' = v''y1+2v'y1'+vy1'' = v'' x^2 +4v'x + 2v
Plug all into the equation under question.
x^2 y'' -3 x y' + 4y = 0 ➔
x^4 v'' + 4 v' x^3 + 2v x^2 – 3v' x^3 - 6v x^2 + 4v x^2 =
x^4 v'' + v' (4 x^3 – 3 x^3] v' + [2x^2 -6 x^2 +4 x^2] v =
x^4 v'' + v'(4 x^3 – 3 x^3) = 0 ➔ v'' + 1/x v'= 0;

Substitute u=v'; u'=v'';

du/dx + u/x = 0 eventually ln|u| = -ln|x|+c ➔ln|u|=-ln|x|+ln|c| = u = C/x
v(x) = int{u(x)}dx ➔ v(x) = int{C/x}dx = C ln(x) + C2

y2= Cx^2 ln|x| + C2 x^2

This is also general equation to the question, since the second part is the solution y1=x^2,

3. Apply derivation operator on y to solve $(D-r_1)(D-r_2)\, y = 0$, hint reduce equation to the first order by substitution.

Answer:

(D-r2)y = u;

(D-r1)u = 0 gives u=c1e^(r1 t), rearranging equation offers a Bernoulli structure

(D-r2)y = c1 e^(r1 t), which is solved

d[e^(-r2 t) y] = c1 e^(r1 t) e^(-r2 t), .

e^(-r2 t) y = int{ c1 e^(r1 t) e^(-r2 t)}dt+c2

y = {c1 e^(r2 t) * e^(r1-r2)t }/(r1-r2)+ c2 e^(r2 t)

y = {c1/(r1-r2}) e^(r21t) + c2 e^(r2 t)

4. $y'' - 3y' + 2y = e^x$

Answers:

a) Solve the reduced (homogeneous) version of this DE.

yh=c1 e^x + c2 e^2x;

b) Solve particular solution of non-homogeneous DE by using the method of undetermined coefficients.

yp=Ae^x vanishes, therefore next possible candidate solution must be in the form of for yp is Axe^x,

yp' = Ae^x+Axe^x = A(1+x)e^x; yp''= Ae^x + A(1+x)e^x = A(2+x)e^x

A(2+x)e^x - 3A(1+x)e^x +2 Axe^x = e^x

A=-1;

c) Solve the same particular solution by using ESL.

p(D) = D^2-3D+2, p(D)' = 2D-3

RHS e^x, where alpha is 1,

since p(alpha)=0, p(alpha)'= -1

yp=xe^x/p(alpha') = -xe^x;

5. Find the particular solution of DE: $y^{(4)} + 5y'' + 2y = 2 - 5e^{(3x)}$ simply by using ESL.

Hint: represent the equation in the form of operators and then apply ESL step by step.

Hints: write as operator, and then apply ESL

p(D) = D^4+5*D^2+2

e^0x ➔ alpha = 0, and p(D) = 2; 2/p(alpha)➔1

e^3x ➔ alpha = 3, and p(D) = 81+45+2 = 128 ➔ e^3x/128

From the superposition of both solutions, 1-5e^(3x)/128