4.3. Subspaces

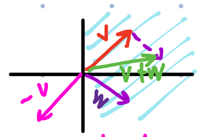
20.04.2020

V = vector space

$W \subset V$ and W is also a v.space $\Rightarrow W$ a subspace.

THEOREM. W is a subspace if (a) $\forall \vec{v}, \vec{w} \in W \quad \vec{v}+\vec{w} \in W$

(b) $\forall c \in \mathbb{R}, \forall \vec{v} \in W, \quad c\vec{v} \in W$

ex 

$V = \mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \right\}$, $W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b \geq 0 \right\}$

Is W a subspace of V?

$-v \notin W$

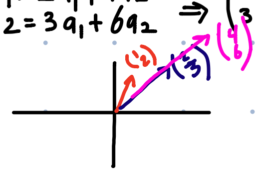
No. W is not a subspace.

$v_1, v_2, \dots, v_k \in V$ and $a_1, \dots, a_k \in \mathbb{R} \Rightarrow a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$ linear combination

ex Is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ a lin. comb. of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$?

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ has a solution a_1, a_2?

$\begin{aligned} 1 &= 2a_1 + 4a_2 \\ 2 &= 3a_1 + 6a_2 \end{aligned} \Rightarrow \left(\begin{array}{cc|c} 2 & 4 & 1 \\ 3 & 6 & 2 \end{array}\right) \Rightarrow -\frac{3}{2} r_1 + r_2 \to r_2 \quad \left(\begin{array}{cc|c} 2 & 4 & 1 \\ 0 & 0 & 1/2 \end{array}\right) \Rightarrow 0 \cdot a_1 + 0 \cdot a_2 = \frac{1}{2}$. No sol.



THM $A_{m \times n}$ matrix. The solution set of $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

proof. If $\vec{x}_1, \vec{x}_2$ are solutions $\Rightarrow A(\vec{x}_1 + \vec{x}_2) = \vec{0}$? YES.

$A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$

If $\vec{x}_1$ is a soln. and $c \in \mathbb{R} \Rightarrow A(c\vec{x}_1) = \vec{0}$? YES.

$c(A\vec{x}_1) = c\vec{0} = \vec{0}$

ex. $\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \Rightarrow$ 2 eqs. 3 unknowns. $x_3 = r$ = free, $x_2 = -r$, $x_1 = r$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ -r \\ r \end{bmatrix} \to$ SOLN. SET. $\to$ SUBSPACE of $\mathbb{R}^3$.

4.4. SPAN.

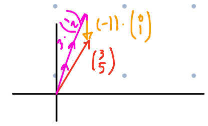
V = v.space. $v_1, v_2, \dots, v_k \in V$. The set of all possible lin. comb. of $v_1, \dots, v_k$ is called the span of $v_1, \dots, v_k$.

ex. Find the span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$.

$=$ all lin. comb. $a_1\begin{pmatrix}1\\2\end{pmatrix}+a_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}a_1\\2a_1+a_2\end{pmatrix}$. Any vector can be written like this.

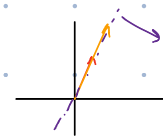
$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a_1\\2a_1+a_2\end{pmatrix}\Rightarrow a_1=x,\ a_2=y-2x$ ex. $\begin{pmatrix}3\\5\end{pmatrix}=3\begin{pmatrix}1\\2\end{pmatrix}+(-1)\begin{pmatrix}0\\1\end{pmatrix}$

The span of $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ is $\mathbb{R}^2$.



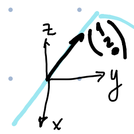
ex Find the span of $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\4\end{pmatrix}$.

$=$ all lin. comb. of $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\4\end{pmatrix}$. $a_1\begin{pmatrix}1\\2\end{pmatrix}+a_2\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}a_1+2a_2\\2a_1+4a_2\end{pmatrix}=(a_1+2a_2)\begin{pmatrix}1\\2\end{pmatrix}$



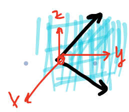
lin. comb. is a line in the direction $\begin{pmatrix}1\\2\end{pmatrix}$.

ex $\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\4\\0\end{pmatrix}$. Span?



span is a line.

ex $\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\3\end{pmatrix}$



span of $\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\\3\end{pmatrix}$ is $a_1\begin{pmatrix}1\\2\\0\end{pmatrix}+a_2\begin{pmatrix}0\\2\\3\end{pmatrix}=\begin{pmatrix}a_1\\2a_1+2a_2\\3a_2\end{pmatrix}$

span is all vectors of this form.

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\2a_1+2a_2\\3a_2\end{pmatrix}\Rightarrow \begin{matrix}a_1=x\\ a_2=\frac{z}{3}\end{matrix},\ y=2x+\frac{2z}{3}$

$\begin{pmatrix}1\\0\\3\end{pmatrix}$ is in the span?

$\begin{matrix}a_1=1\\ a_2=1\end{matrix}$ $0\neq 2+2$. No, it is not.

ex $V=M_{2\times 2}$ $\vec{v}_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ \vec{v}_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ \vec{v}_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}$

Span of $v_1,v_2,v_3\to\begin{pmatrix}a&b\\c&0\end{pmatrix}=a\vec{v}_1+b\vec{v}_2+c\vec{v}_3$

THM. Span$\{v_1,\dots,v_k\}$ is a subspace of V.

ex $V=P_2=$ v.space of polynomials with degree $\leqslant 2$.

$S=\{\underset{v_1}{t^2},\underset{v_2}{t}\}\Rightarrow$ span $S=$ all polys of the form $at+bt^2$.

ex Is t^2-t in the span of t^2+1 and $t-1$?

$t^2-t=a(t^2+1)+b(t-1)=t^2a+tb+(a-b)\Rightarrow a=1,\ b=-1$, $a-b=0$. No soln.

ex $v_1=\begin{bmatrix}2\\1\\1\end{bmatrix},\ v_2=\begin{bmatrix}1\\-1\\3\end{bmatrix}$ $v=\begin{bmatrix}1\\5\\-7\end{bmatrix}$. Is v in the span of v_1,v_2?

Is there a soln. of $v=av_1+bv_2$ for a,b?

$\begin{bmatrix}1\\5\\-7\end{bmatrix}=a\begin{bmatrix}2\\1\\1\end{bmatrix}+b\begin{bmatrix}1\\-1\\3\end{bmatrix}\Rightarrow\begin{matrix}2a+b=1\\a-b=5\\a+3b=-7\end{matrix}\Rightarrow\left[\begin{array}{cc|c}2&1&1\\1&-1&5\\1&3&-7\end{array}\right]\xrightarrow{r_1\leftrightarrow r_2}\left[\begin{array}{cc|c}1&-1&5\\2&1&1\\1&3&-7\end{array}\right]\begin{matrix}-2r_1+r_2\to r_2\\-r_1+r_3\to r_3\end{matrix}\left[\begin{array}{cc|c}1&-1&5\\0&3&-9\\0&4&-12\end{array}\right]\begin{matrix}\frac{r_2}{3}\to r_2\\ r_3/4\to r_3\end{matrix}\left[\begin{array}{cc|c}1&-1&5\\0&1&-3\\0&1&-3\end{array}\right]-r_2+r_3\to r_3$

YES. $\left[\begin{array}{cc|c}1&-1&5\\0&1&-3\\0&0&0\end{array}\right]$ $\begin{matrix}b=-3\\a-b=5\\a=2.\end{matrix}$

ex $v_1=\begin{bmatrix}1\\2\\1\end{bmatrix},\ v_2=\begin{bmatrix}1\\0\\2\end{bmatrix},\ v_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$. Do these vectors span $\mathbb{R}^3$?

Can I write $\begin{bmatrix}x\\y\\z\end{bmatrix}=av_1+bv_2+cv_3$? $\begin{matrix}x=a+b+c\\y=2a+0b+c\\z=a+2b+0c\end{matrix}$. Can I solve for a,b,c in terms of x,y,z?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 2 & 0 & 1 & y \\ 1 & 2 & 0 & z \end{array}\right] \begin{array}{l} -2r_1+r_2\to r_2 \\ -r_1+r_3\to r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -2 & -1 & y-2x \\ 0 & 1 & -1 & z-x \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & -1 & z-x \\ 0 & -2 & -1 & y-2x \end{array}\right] \xrightarrow{2r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & -1 & z-x \\ 0 & 0 & -3 & y-2x+2(z-x) \end{array}\right] \quad c=\frac{y-2x+2(z-x)}{-3}$$

The answer is yes.

<u>ex</u> $V=P_2$. $v_1=t^2+2t+1$, $v_2=t^2+2$. Does v_1,v_2 span P_2?

$a+bt+ct^2 = a_1v_1+a_2v_2$. Can we solve for a_1,a_2?

$$= a_1(t^2+2t+1)+a_2(t^2+2) = t^2(a_1+a_2)+t(2a_1)+a_1+2a_2$$

$$\begin{array}{l} a_1+a_2=c \\ 2a_1=b \\ a_1+2a_2=a \end{array} \Rightarrow \begin{array}{l} a_1=b/2 \\ a_2=c-b/2 \end{array} \Rightarrow \boxed{\frac{b}{2}+2\left(c-\frac{b}{2}\right)=a} \Rightarrow 2c-\frac{b}{2}=a$$

For example $a=1$, $b=2$, $c=1$ $\Rightarrow$ $1+2t+t^2$ is in the span.

But $a=1$, $b=2$, $c=4$ $\Rightarrow$ $1+2t+4t^2$ is not in the span. The answer is NO!

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ex
$$A=\begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}$$
$A\vec{x}=\vec{0}$. Find a set of vectors which span the solution space.

$$\left[\begin{array}{c|c} A & \begin{matrix}0\\0\\0\\0\end{matrix} \end{array}\right] \begin{array}{l} 2r_1+r_2\to r_2 \\ -r_1+r_3\to r_3 \\ -4r_1+r_4\to r_4 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} r_2+r_3\to r_3 \\ r_2+r_4\to r_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1+x_2+2x_4=0$, $x_3-x_4=0$, $\begin{array}{l} x_2=r \\ x_4=s \end{array}$ (free) $\begin{array}{l} x_3=s \\ x_1=-r-2s \end{array}$

$$\vec{x}=\begin{bmatrix} x_1\\x_2\\x_3\\x_4 \end{bmatrix}=\begin{bmatrix} -r-2s\\r\\s\\s \end{bmatrix}=\begin{bmatrix} -r\\r\\0\\0 \end{bmatrix}+\begin{bmatrix} -2s\\0\\s\\s \end{bmatrix}=r\begin{bmatrix} -1\\1\\0\\0 \end{bmatrix}+s\begin{bmatrix} -2\\0\\1\\1 \end{bmatrix} \quad \text{Sol. Sp.} = \text{Span}\left\{\begin{bmatrix} -1\\1\\0\\0 \end{bmatrix},\begin{bmatrix} -2\\0\\1\\1 \end{bmatrix}\right\}$$

4.5. LINEAR INDEPENDENCE

$W=\left\{\begin{bmatrix} a\\b\\a+b \end{bmatrix}\right\} \subseteq \mathbb{R}^3$ subspace.

$$W=\text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$$

$V=$ v. space, $\vec{v}_1,\dots,\vec{v}_k \in V$ **linearly dependent** if there exists scalars $a_1,\dots,a_k$ not all zero

$$a_1\vec{v}_1+a_2\vec{v}_2+\dots+a_k\vec{v}_k=\vec{0}$$

Nontrivial sol.

If $\{v_1,\dots,v_k\}$ is not lin. dep. $\Rightarrow$ **lin. independent.**

only trivial soln.

<u>ex</u> Is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ lin. dep. or indep.?

Sol. $a_1\begin{bmatrix}1\\0\\1\end{bmatrix}+a_2\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{l} a_1=0 \\ a_2=0 \\ a_1+a_2=0 \end{array}$. Only 1 sol. $a_1=a_2=0$. Trivial sol. $\Rightarrow$ lin. indep.

<u>ex</u> Is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$ lin. dep. or indep.?

sol. $a_1\begin{bmatrix}1\\0\\1\end{bmatrix}+a_2\begin{bmatrix}0\\1\\1\end{bmatrix}+a_3\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}a_1+a_3=0\\a_2+a_3=0\\a_1+a_2+2a_3=0\end{matrix} \quad \left[\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\1&1&2&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&1&1&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&0&0\end{array}\right]$

$a_3=r=\text{free}$
$a_2=-r$
$a_1=-r$
$\Rightarrow \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}-r\\-r\\r\end{bmatrix} \Rightarrow$ Nontrivial solns $\Rightarrow$ lin. dep.

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ex. $v_1=[1\ \ 0\ \ 0]$, $v_2=[0\ \ 1\ \ 0]$ $v_3=[0\ \ 0\ \ 1]$. Are these vectors lin. dep. or indep?

Sol. $a_1\vec{v}_1+a_2\vec{v}_2+a_3\vec{v}_3=\vec{0} \Rightarrow a_1[1\ 0\ 0]+a_2[0\ 1\ 0]+a_3[0\ 0\ 1]=[0\ 0\ 0]$

$[a_1\ \ a_2\ \ a_3]=[0\ \ 0\ \ 0] \Rightarrow a_1=a_2=a_3=0$. Lin. indep.

ex. $V=P_2$. $v_1=t^2+t+2$, $v_2=2t^2+t$, $v_3=3t^2+2t+2$. Lin. dep or lin. indep?

Sol. $a_1(t^2+t+2)+a_2(2t^2+t)+a_3(3t^2+2t+2)=0$

1st method

$t^2\underbrace{(a_1+2a_2+3a_3)}_{0}+t\underbrace{(a_1+a_2+2a_3)}_{0}+\underbrace{(2a_1+2a_3)}_{0}=0$

Recall $A_{n\times n}$ $A\vec{x}=\vec{0}$: $\det A=0$ → nontrivial sol. ; $\det A\neq 0$ → only the trivial solution.

$\begin{matrix}a_1+2a_2+3a_3=0\\a_1+a_2+2a_3=0\\2a_1+2a_3=0\end{matrix} \Rightarrow \begin{vmatrix}1&2&3\\1&1&2\\2&0&2\end{vmatrix}=(-1)^{3+1}2\begin{vmatrix}2&3\\1&2\end{vmatrix}+0+(-1)^{3+3}2\begin{vmatrix}1&2\\1&1\end{vmatrix}=2\cdot(4-3)+2(1-2)=2-2=0.$

Nontrivial sol. $\Rightarrow$ lin. dep.

2nd method. $v_1+v_2=v_3 \Rightarrow 1v_1+1v_2-1v_3=0 \Rightarrow$ lin. dep.

thm. $v_1,\dots,v_k$ in $\mathbb{R}^k$ are lin. indep. $\Leftrightarrow \det([v_1\cdots v_k])\neq 0$

lin. dep $\Leftrightarrow \det([v_1\cdots v_k])=0$

ex. $S=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix},\begin{bmatrix}3\\0\\-1\end{bmatrix}\right\}$. Lin. dep. or lin. indep.?

Sol. Check $\begin{vmatrix}1&0&3\\2&1&0\\3&2&-1\end{vmatrix}=2 \Rightarrow$ lin. indep.

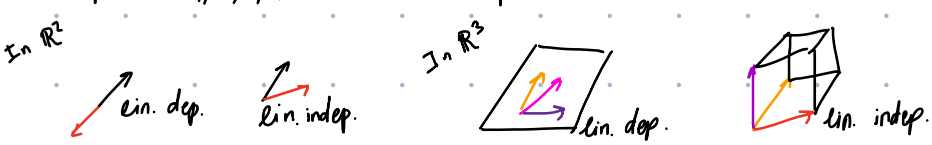
thm. If at least one of $v_1,\dots,v_k$ is the zero-vec. then $v_1,\dots,v_k$ is lin. dep.

thm. $V=$ v. space. $S_1\subset S_2\subset V$. If S_1 is lin. dep. $\Rightarrow S_2$ is lin. dep.

If S_2 is lin. indep. $\Rightarrow S_1$ is lin. indep.

ex v_1,v_2,v_3 is lin. dep. $\Rightarrow v_1,v_2,v_3,v_4$ is also lin. dep.

Geometric idea



4.6. BASIS AND DIMENSION

def. $V=$v.space. $v_1,\dots,v_k$ in V forms a basis for V if:

a) $v_1,\dots,v_k$ lin. indep. b) span$\{v_1,\dots,v_k\}=V$.

Remark. If $v_1, \dots, v_k$ form a basis $\Rightarrow$ they must be distinct and non-zero.

ex. For $\mathbb{R}^2$ $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ form a standard basis.

a) Lin. indep. $a_1\begin{bmatrix}1\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow a_1 = a_2 = 0 \Rightarrow$ lin. indep.

b) $\text{span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = \mathbb{R}^2 \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$

ex For $\mathbb{R}^2$ $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$ form a basis for $\mathbb{R}^2$.

a) Lin. indep $\Rightarrow \begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} = 1 \neq 0$

b) $\begin{bmatrix}x\\y\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}$ $\quad \begin{matrix} x = a+b \\ y = 0+b \end{matrix} \Rightarrow \begin{matrix} b = y \\ a = x - b = x - y. \end{matrix}$

ex. For $\mathbb{R}^3$ $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ form a standard basis.

ex $V = P_2$. $S = \{t^2+1,\ t-1,\ 2t+2\}$. Is S a basis for P_2?

sol a) lin. indep. $a(t^2+1) + b(t-1) + c(2t+2) = 0$

$$a t^2 + (b+2c)t + (a-b+2c) = 0 \Rightarrow \left.\begin{matrix} a=0 \\ b+2c=0 \\ a-b+2c=0 \end{matrix}\right\} \Rightarrow \begin{matrix} c=0 \\ b=0 \end{matrix}$$ Only Trivial Soln. $\Rightarrow$ Lin. indep.

b) span $S = P_2$? Yes!

$$at^2 + bt + c = a_1(t^2+1) + a_2(t-1) + a_3(2t+2)$$

$\boxed{a_1 = a}$, $a_2 + 2a_3 = b$, $a_1 - a_2 + 2a_3 = c$ $\Rightarrow$ $\begin{matrix} a_2 + 2a_3 = b \\ -a_2 + 2a_3 = c - a \end{matrix} \Rightarrow 4a_3 = b + c - a$, $2a_2 = b - c + a$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\1 & -1 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

Recall. $A_{n\times n}$ $\quad A\vec{x} = \vec{b}$. If A^{-1} exists $\vec{x} = A^{-1}\vec{b}$.

$$\Updownarrow$$

$$\det A \neq 0$$

For the prev. problem. It is sufficient to check $\det\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\1 & -1 & 2\end{bmatrix}\right) \neq 0 \Rightarrow$ Basis!

ex. $V = P_2$. $\quad W \subset V$, $W = \{at^2 + bt + (a-b)\}$ subspace V. ex/ $2t^2 + 3t - 1 \in W$, $3t^2 + 1 \notin W$

Find a basis for W?

sol $at^2 + bt + (a-b) = a\underbrace{(t^2+1)} + b\underbrace{(t-1)}$. Choose $S = \{t^2+1,\ t-1\}$. Show S is a basis for W.

a) lin. indep. $a(t^2+1) + b(t-1) = 0 \Rightarrow \begin{matrix} a=0 \\ b=0 \end{matrix}$ only trivial solution $\Rightarrow$ lin. indep.

b) $\text{span}(S) = W$? ✓.

def. V = v.space. and let S be a basis of V.

if S has ∞ vectors $\Rightarrow V = \infty$ dimensional v.space.

S has finitely many vectors $\Rightarrow$ V = finite dim. v.space.

Number of vectors in S is called the **dimension** of V.

Axiom of choice $\Rightarrow$ Any v.space has a basis.

The number of vectors in S is independent of S.

ex $v_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $v_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $v_3=\begin{bmatrix}1\\1\\2\end{bmatrix}$, $v_4=\begin{bmatrix}1\\2\\1\end{bmatrix}$, $v_5=\begin{bmatrix}-1\\1\\-2\end{bmatrix}$

$S=\{v_1,\dots,v_5\}$. Find a basis for span(S).

sol. $a_1\vec{v}_1+\dots+a_5\vec{v}_5=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{ccccc|c}1&0&1&1&-1&0\\0&1&1&2&1&0\\1&1&2&1&-2&0\end{array}\right] \xRightarrow{RREF} \begin{array}{c}\left[\begin{array}{ccccc|c}1&0&1&0&-2&0\\0&1&1&0&-1&0\\0&0&0&1&1&0\end{array}\right]\\ \begin{array}{ccccc}a_1&a_2&a_3&a_4&a_5\end{array}\end{array}$ $a_3=r$, $a_5=s$, $a_4=-s$, $a_1=-r+2s$, $a_2=-r+s$

$\text{span}(S)=\{v_1,v_2,v_4\} \rightarrow \boxed{\text{basis}}$ Suppose we remove $v_3,v_5 \Rightarrow$ RREF $\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$

span

We choose vectors which have a leading 1.

28.04

(cont.) $(-r+2s)\vec{v}_1+(-r+s)\vec{v}_2+r\vec{v}_3-s\vec{v}_4+s\vec{v}_5=\vec{0}$

$r(-\vec{v}_1-\vec{v}_2+\vec{v}_3)+s(2\vec{v}_1+\vec{v}_2-\vec{v}_4+\vec{v}_5)=\vec{0}$ for every r,s.

$s=0,\ r=1 \Rightarrow -\vec{v}_1-\vec{v}_2+\vec{v}_3=\vec{0} \Rightarrow \vec{v}_3=\vec{v}_1+\vec{v}_2$

$r=0,\ s=1 \Rightarrow \vec{v}_5=-2\vec{v}_1-\vec{v}_2+\vec{v}_4$

thm. If $\vec{v}=a_1\vec{v}_1+\dots+a_k\vec{v}_k$, then $\text{span}\{\vec{v},\vec{v}_1,\dots,\vec{v}_k\}=\text{span}\{\vec{v}_1,\dots,\vec{v}_k\}$.

ex. $V=\mathbb{R}_3$. $S=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$. $\vec{v}_1=[0\ 1\ 1]$, $\vec{v}_2=[1\ 0\ 1]$ $\vec{v}_3=[1\ 1\ 2]$

Find a basis for span$S$. Find $\dim(\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\})$

sol. Observe: $\vec{v}_3=\vec{v}_1+\vec{v}_2$. $\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}=\text{span}\{\vec{v}_1,\vec{v}_2\}$

We are done if $\{\vec{v}_1,\vec{v}_2\}$ is lin. indep.

$a_1\vec{v}_1+a_2\vec{v}_2=\vec{0} \Rightarrow a_1[0\ 1\ 1]+a_2[1\ 0\ 1]=[0\ 0\ 0] \Rightarrow [a_2, a_1, a_1+a_2]=[0\ 0\ 0]$

$a_1=a_2=0$. Only the trivial sol. exists.

So $\{\vec{v}_1,\vec{v}_2\}$ are both lin. indep. and span $\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$. So $\{\vec{v}_1,\vec{v}_2\}$ is a basis for $\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

$\dim(\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\})=2.$

thm. $\dim V=n. \Rightarrow$ any subset S of V with more than n vectors must be lin. dep.

$\Rightarrow$ any subset ... with fewer than ...

can not span V.

ex Find a basis for P_3.

sol $a_3t^3+a_2t^2+a_1t+a_0 1 \Rightarrow P_3 = \text{span}\{1, t, t^2, t^3\}$. Moreover, they are lin. indep.

$a_0+a_1t+a_2t^2+a_3t^3=0 \Leftrightarrow a_0=a_1=a_2=a_3=0$

$\dim(P_3)=4$. $\dim(P_n)=n+1$.

ex Is $\{t^2-1, t+1, t-1, t^2+t\}$ a basis for P_2?

sol No. More than 3 vectors in P_2 must be lin. dep.

4.7. Homogeneous Systems.

$A_{m\times n}$ matrix. $A\vec{x}=\vec{0} \Leftrightarrow$ Homogeneous sys.

thm The solution set of $A\vec{x}=\vec{0}$ is a subspace of $\mathbb{R}^n$.

proof If $\vec{x}_1$ and $\vec{x}_2$ are solutions, $A\vec{x}_1=\vec{0}$, $A\vec{x}_2=\vec{0} \Rightarrow A(\vec{x}_1+\vec{x}_2)=A\vec{x}_1+A\vec{x}_2=\vec{0}+\vec{0}=\vec{0}$.

$c\in\mathbb{R}$, $\vec{x}$ is a soln. $\Rightarrow A(c\vec{x})=cA\vec{x}=c\vec{0}=\vec{0}$.

def. Sol. space of $A\vec{x}=\vec{0}$ is called nullspace of A and dimension of nullspace is called nullity.

ex Suppose that RREF of $A\vec{x}=\vec{0}$. $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Find a basis for the sol. sp.

$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ $\quad x_3=r$, $x_5=s$ $\quad x_4+2x_5=0$, $x_4=-2x_5=-2s$ $\quad x_1=-2x_3-x_5=-2r-s$ $\quad x_2=-2x_3+x_5=-2r+s$

$\vec{x}=\begin{bmatrix} -2r-s \\ -2r+s \\ r \\ -2s \\ s \end{bmatrix} = r\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ form a basis for the sol. sp.

Nullity of $A=2$.

remark The solution set of $A\vec{x}=\vec{b}$, $\vec{b}\neq\vec{0}$ is not a vector space. Because

if $A\vec{x}_1=\vec{b}$, $A\vec{x}_2=\vec{b} \Rightarrow A(\vec{x}_1+\vec{x}_2)=2\vec{b}\neq\vec{b}$

Any solution of $A\vec{x}=\vec{b}$ can be written as a sum of a soln. of $A\vec{x}=\vec{0}$ and one particular soln. of $A\vec{x}=\vec{b}$.

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4.8. COORDINATES.

V = v.space. An ordered basis $S=\{\vec{v}_1,\dots,\vec{v}_n\}$.

$\vec{v}\in V \Rightarrow \vec{v}=a_1\vec{v}_1+\dots+a_n\vec{v}_n$, $a_1,\dots,a_n\in\mathbb{R}$.

$[\vec{v}]_S=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$ = coordinates of $\vec{v}$ w.r.t. S.

ex $V=\mathbb{R}^2$. $S=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ basis.

$\vec{v}=\begin{bmatrix}3\\2\end{bmatrix}$. $[\vec{v}]_S=?$

solution. $\begin{bmatrix}3\\2\end{bmatrix}=a_1\begin{bmatrix}1\\0\end{bmatrix}+a_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}a_1+a_2\\a_2\end{bmatrix}\Rightarrow \begin{matrix}a_2=2\\a_1=1\end{matrix}\Rightarrow [\vec{v}]_S=\begin{bmatrix}1\\2\end{bmatrix}$. To obtain $\begin{bmatrix}3\\2\end{bmatrix}$,

ex $S=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$. $\begin{bmatrix}3\\2\end{bmatrix}_S=\begin{bmatrix}2\\1\end{bmatrix}$

$S=\{v_1,\dots,v_n\}$.

$[\vec{v}]_S=\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$, $[\vec{w}]_S=\begin{bmatrix}b_1\\\vdots\\b_n\end{bmatrix}$. $[\vec{v}+\vec{w}]_S=\begin{bmatrix}a_1+b_1\\\vdots\\a_n+b_n\end{bmatrix}$. Why?

$\vec{v}=a_1\vec{v}_1+\dots+a_n\vec{v}_n$, $\vec{w}=b_1\vec{v}_1+\dots+b_n\vec{v}_n$. $\vec{v}+\vec{w}=(a_1+b_1)\vec{v}_1+\dots+(a_n+b_n)\vec{v}_n$

$c\in\mathbb{R}$ $[c\vec{v}]_S=c[\vec{v}]_S$. proof similar.

Transition Matrices.

$S=\{\vec{v}_1,\dots,\vec{v}_n\}$, $T=\{\vec{w}_1,\dots,\vec{w}_n\}$ ordered bases

$\vec{v}=c_1\vec{w}_1+\dots+c_n\vec{w}_n \Rightarrow [\vec{v}]_T=\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}$

$[\vec{v}]_S=[c_1\vec{w}_1+\dots+c_n\vec{w}_n]_S=c_1[\vec{w}_1]_S+\dots+c_n[\vec{w}_n]_S=\begin{bmatrix}[w_1]_S & [w_2]_S & \cdots & [w_n]_S\end{bmatrix}\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$.

$=P_{S\leftarrow T}[v]_T$

$\hookrightarrow$ transition matrix from the T-basis to S-basis.

Reminder

ex $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $\vec{x}=\begin{bmatrix}y\\z\end{bmatrix}$

$A\vec{x}=\begin{bmatrix}ay+bz\\cy+dz\end{bmatrix}=y\begin{bmatrix}a\\c\end{bmatrix}+z\begin{bmatrix}b\\d\end{bmatrix}$

ex. $V=\mathbb{R}^2$. $S=\left\{\underset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}},\underset{v_2}{\begin{bmatrix}1\\1\end{bmatrix}}\right\}$. $T=\left\{\underset{w_1}{\begin{bmatrix}0\\1\end{bmatrix}},\underset{w_2}{\begin{bmatrix}-1\\0\end{bmatrix}}\right\}$ (a) $P_{S\leftarrow T}=?$

$w_1=c_1\vec{v}_1+c_2\vec{v}_2\Rightarrow\begin{bmatrix}0\\1\end{bmatrix}=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}c_1+c_2\\c_2\end{bmatrix}\Rightarrow\begin{matrix}c_2=1\\c_1=-1\end{matrix}$ $[\vec{w}_1]_S=\begin{bmatrix}-1\\1\end{bmatrix}$

$\vec{w}_2=c_1\vec{v}_1+c_2\vec{v}_2\Rightarrow\begin{bmatrix}-1\\0\end{bmatrix}=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}c_1+c_2\\c_2\end{bmatrix}\Rightarrow\begin{matrix}c_2=0\\c_1=-1\end{matrix}\Rightarrow[\vec{w}_2]_S=\begin{bmatrix}-1\\0\end{bmatrix}$

$P_{S\leftarrow T}=\begin{bmatrix}-1&-1\\1&0\end{bmatrix}$.

(b) If $[v]_T=\begin{bmatrix}2\\3\end{bmatrix}\Rightarrow[v]_S=?$

$[\vec{v}]_S=P_{S\leftarrow T}[\vec{v}]_T=\begin{bmatrix}-1&1\\1&0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

ex/ $V = P_1$ = polys of deg ≤ 1.

$\vec{v}_1 = t$, $\vec{v}_2 = t-3$, $\vec{w}_1 = t-1$, $\vec{w}_2 = t+1$.

$S = \{\vec{v}_1, \vec{v}_2\}$, $T = \{\vec{w}_1, \vec{w}_2\}$.

a) Find $P_{S\leftarrow T}$

$[w_1]_S = ? \Leftrightarrow \vec{w}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 \Leftrightarrow t-1 = c_1 t + c_2(t-3) = t(c_1+c_2) - 3c_2 \Rightarrow c_2 = 1/3 \quad c_1 = 2/3$

$[w_1]_S = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

$[w_2]_S = ? \Leftrightarrow t+1 = c_1 t + c_2(t-3) = t(c_1+c_2) - 3c_2 \Rightarrow \begin{matrix} c_2 = -1/3 \\ c_1 = 4/3 \end{matrix}$

$[w_2]_S = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$

$P_{S\leftarrow T} = \begin{bmatrix} [w_1]_S & [w_2]_S \end{bmatrix} = \begin{bmatrix} 2/3 & 4/3 \\ 1/3 & -1/3 \end{bmatrix}$

(b) $v = 5t+1$. Find $[\vec{v}]_T$, $[\vec{v}]_S$.

$5t+1 = c_1 w_1 + c_2 w_2 = c_1(t-1) + c_2(t+1) \Rightarrow \begin{matrix} c_1+c_2=5 \\ c_2-c_1=1 \end{matrix} \Rightarrow \begin{matrix} c_2=3 \\ c_1=2 \end{matrix} \Rightarrow [5t+1]_T = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$5t+1 = c_1 v_1 + c_2 v_2 = c_1 t + c_2(t-3) \Rightarrow \begin{matrix} c_1+c_2=5 \\ -3c_2=1 \end{matrix} \Rightarrow \begin{matrix} c_2=-1/3 \\ c_1=16/3 \end{matrix} \Rightarrow [5t+1]_S = \begin{bmatrix} 16/3 \\ -1/3 \end{bmatrix}$

(c) Verify $[\vec{v}]_S = P_{S\leftarrow T}[\vec{v}]_T$. $\quad [\vec{v}]_S = \begin{bmatrix} 2/3 & 4/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/3+12/3 \\ 2/3-3/3 \end{bmatrix} = \begin{bmatrix} 16/3 \\ -1/3 \end{bmatrix}$ ✓

$Q_{T\leftarrow S}$ = transition matrix from $S \to T$

$P_{S\leftarrow T}$ = trans. mat. from $T \to S$

$Q_{T\leftarrow S} = (P_{S\leftarrow T})^{-1}$

Read: 4.9. Rank of a matrix.

CHAPTER 6.

6.1. Linear Transformations.

V, W = v. spaces. $L: V \to W$ is called a **linear transformation** from V to W if

(a) $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v})$, $\forall \vec{u},\vec{v} \in V$

(b) $L(c\vec{u}) = cL(\vec{u})$, $\forall c \in \mathbb{R}$, $\forall \vec{u} \in V$

ex/ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ $\quad L: \mathbb{R}^2 \to \mathbb{R}^2 \quad L(\vec{u}) = A\vec{u}$. Is L, a lin. trans.?

Yes. $L(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = L(\vec{u}) + L(\vec{v})$ ✓

$L(c\vec{u}) = A(c\vec{u}) = c\,A\vec{u} = cL(\vec{u})$ ✓

ex $A_{m\times n}$ matrix $L(\vec{u}) = A\vec{u}$, $L: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. trans.

ex. $L: P_2 \to \mathbb{R}^2$, $L(a_0+a_1t+a_2t^2) = \begin{bmatrix} a_1 \\ a_0+a_2 \end{bmatrix}$. Is this a lin. trans.? YES!

a) $L(a_0+a_1t+a_2t^2 + b_0+b_1t+b_2t^2) = L((a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2) = \begin{bmatrix} a_1+b_1 \\ a_0+b_0+a_2+b_2 \end{bmatrix}$

$L(a_0+a_1t+a_2t^2) + L(b_0+b_1t+b_2t^2) = \begin{bmatrix} a_1 \\ a_0+a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_0+b_2 \end{bmatrix}$ YES!

b) $L(c(a_0+a_1t+a_2t^2)) = L(ca_0+ca_1t+ca_2t^2) = \begin{bmatrix} ca_1 \\ ca_0+ca_2 \end{bmatrix}$

$cL(a_0+a_1t+a_2t^2) = c\begin{bmatrix} a_1 \\ a_0+a_2 \end{bmatrix}$ YES!

ex $L: \mathbb{R}^3 \to \mathbb{R}^2$. $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ 2y \end{bmatrix}$. Is L a lin. trans? NO!

a) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L\left(\begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}\right) = \begin{bmatrix} x+a+1 \\ 2(y+b) \end{bmatrix}$ not equal.

$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ 2y \end{bmatrix} + \begin{bmatrix} a+1 \\ 2b \end{bmatrix} = \begin{bmatrix} x+a+2 \\ 2y+2b \end{bmatrix}$

$L(\vec{u}+\vec{v}) \neq L(\vec{u}) + L(\vec{v})$.

ex $L: \mathbb{R}_2 \to \mathbb{R}_2$. $L([x \ \ y]) = [x^2 \ \ y]$. Is L a lin. trans? No!

$L([1 \ \ 0]) + L([2 \ \ 0]) \stackrel{?}{=} L([3 \ \ 0])$ Not a lin. trans.

$[1 \ \ 0] + [4 \ \ 0] = [5 \ \ 0] \neq [9 \ \ 0]$

ex $L: P_1 \to P_2$ $L[p(t)] = t\,p(t)$. Is L a lin. trans? Yes!

a) $L[p(t)+q(t)] = t(p(t)+q(t)) = tp(t)+tq(t) = L[p(t)] + L[q(t)]$ ✓

b) $L[c\,p(t)] = t(cp(t)) = c(t\,p(t)) = cL[p(t)]$ ✓

Thm $L: V \to W$ is a lin. trans. Then:

a) $L(\vec{0}_V) = \vec{0}_W$ proof. $L(\vec{0}_V) = L(\vec{0}_V+\vec{0}_V) = L(\vec{0}_V) + L(\vec{0}_V) = 2L(\vec{0}_V) \Rightarrow L(\vec{0}_V) = \vec{0}_W$

b) $L(\vec{u}-\vec{v}) = L(\vec{u}) - L(\vec{v})$.

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ex $L: \mathbb{R}^2 \to \mathbb{R}^3$. $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y+x \\ yx \end{bmatrix}$. Is L a lin. trans?

$L\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow L(0_{\mathbb{R}^2}) \neq 0_{\mathbb{R}^3}$. L is not lin. trans.

ex $L: \mathbb{R}_2 \to \mathbb{R}_2$ lin. trans. $L([1 \ \ 1]) = [1 \ \ {-2}]$, $L([-1 \ \ 1]) = [2 \ \ 3]$.

(a) Find $L([-1 \ \ 5])$

$[-1 \ \ 5] = a[1 \ \ 1] + b[-1 \ \ 1] \Rightarrow \begin{matrix} a-b=-1 \\ a+b=5 \end{matrix} \Rightarrow \begin{matrix} a=2 \\ b=3 \end{matrix}$

$$L([-1\ \ 5]) = L(2[1\ \ 1]+3[-1\ \ 1]) = L(2[1\ \ 1]) + L(3[-1\ \ 1])$$
$$= 2L([1\ \ 1]) + 3L([-1\ \ 1])$$
$$= 2[1\ \ -2]+3[2\ \ 3] = [8\ \ \ 5].$$

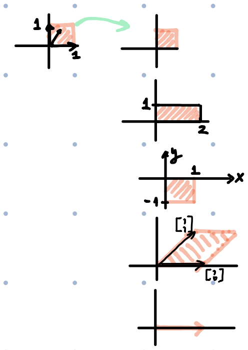
(b) $L([x\ \ y]) = ?$

$$[x\ \ y] = a[1\ \ 1] + b[-1\ \ 1] \Rightarrow \begin{matrix} a-b=x \\ a+b=y \end{matrix} \Rightarrow \begin{matrix} a=(x+y)/2 \\ b=(y-x)/2 \end{matrix}$$

$$L([x\ \ y]) = L\left(\frac{(x+y)}{2}[1\ \ 1] + \frac{(y-x)}{2}[-1\ \ 1]\right) = \frac{(x+y)}{2}L([1\ \ 1]) + \frac{(y-x)}{2}L([-1\ \ 1])$$
$$= \frac{(x+y)}{2}[1\ \ -2] + \frac{(y-x)}{2}[2\ \ 3] = \left[\frac{x+y}{2} + \frac{2y-2x}{2} \quad \frac{-2x-2y+3y-3x}{2}\right] = \left[\frac{-x+3y}{2} \quad \frac{-5x+y}{2}\right]$$

Verify $L([-1\ \ 5]) = \left[\frac{1+15}{2} \quad \frac{5+5}{2}\right] = [8\ \ \ 5]$

St. Mat. Rep.



$L: \mathbb{R}^2 \to \mathbb{R}^2$. $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A = \left[L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\ \ L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \det(A) = 2$

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$L: \mathbb{R}^n \to \mathbb{R}^m$ lin. trans. $\{\vec{e}_1, \dots, \vec{e}_n\}$ standard basis. $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$

Let A be the $m\times n$ matrix whose jth column is $L(e_j)$. Then $L(\vec{x}) = A\vec{x}$ $\forall x \in \mathbb{R}^n$.

A is called the standard matrix representation of L.

ex $L: \mathbb{R}^3 \to \mathbb{R}^2$. $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3y-2z \end{bmatrix}$ Find the st. mat. rep.

$$A = [L(e_1)\ \ L(e_2)\ \ L(e_3)] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

ex. $L: P_1 \to P_1$ lin. trans. $L(t+1) = 2t+3$, $L(t-1) = 3t-2$. Find $L(6t-4)$.

$$6t-4 = a(t+1) + b(t-1) \Rightarrow \begin{matrix} a+b=6 \\ a-b=-4 \end{matrix} \Rightarrow \begin{matrix} a=1 \\ b=5 \end{matrix}$$

$$L(6t-4) = L(1(t+1) + 5(t-1)) = L(t+1) + 5L(t-1) = 2t+3 + 5(3t-2) = 17t+1.$$

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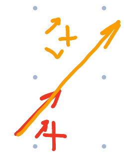
CHAPTER 7 / 7.1. Eigenvalues and eigenvectors



V = v. space. $L: V \to V$ lin. trans.

If there is a vector $\vec{x} \in V$, $\vec{x} \neq \vec{0}_V$ and $\lambda \in \mathbb{C}$ (complex number) such that $L\vec{x} = \lambda\vec{x}$

then $\vec{x}$ is called an eigenvector and λ is called an eigenvalue of L.

For us: $L = A_{n\times n}$ $V = \mathbb{R}^n$. $A\vec{x} = \lambda\vec{x}$

ex $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Find all eigenvalues and eigenvectors.

$A\vec{x} = \lambda\vec{x} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \begin{matrix} x_1 + x_2 = \lambda x_1 \\ -2x_1 + 4x_2 = \lambda x_2 \end{matrix} \Leftrightarrow \begin{matrix} (1-\lambda)x_1 + x_2 = 0 \\ -2x_1 + (4-\lambda)x_2 = 0 \end{matrix} \Leftrightarrow \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The system has non-trivial solutions if and only if $\det\begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix} = 0$ $\to \det(A - \lambda I_2)$

$(1-\lambda)(4-\lambda) + 2 = 0 \Leftrightarrow 6 - 5\lambda + \lambda^2 = 0 \Leftrightarrow \lambda = 2,\ \lambda = 3$. Eigenvalues.

$\lambda = 2$, $\begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{-2r_1 + r_2 \to r_2} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $-x_1 + x_2 = 0$ $\begin{matrix} x_2 = r \\ x_1 = r \end{matrix}$

Eigenvectors of $\lambda = 2$ are of the form $\begin{bmatrix} r \\ r \end{bmatrix}$, $r \neq 0$.

$\lambda = 3$ $\begin{bmatrix} 1-3 & 1 \\ -2 & 4-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow -2x_1 + x_2 = 0$ $\begin{matrix} x_1 = r \\ x_2 = 2r \end{matrix}$

Eigenvectors of $\lambda = 3$ are of the form $\begin{bmatrix} r \\ 2r \end{bmatrix}$, $r \neq 0$.

def. $A_{n\times n}$ matrix. $\det(\underset{n\times n}{cA}) = c^n \det(A)$

$p(\lambda) = \det(\lambda I - A) =$ characteristic polynomial of A.

$\underbrace{p(\lambda) = 0}$ $\Leftrightarrow$ eigenvalues

characteristic equation.

ex $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Find the char. poly. of A.

$p(\lambda) = \det(\lambda I_3 - A)$

$\lambda I_3 - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda-5 \end{bmatrix}$

$\det(\lambda I - A) = (-1)^{1+1}(\lambda-1)\begin{vmatrix} \lambda & -1 \\ 4 & \lambda-5 \end{vmatrix} + (-1)^{1+2}(-2)\begin{vmatrix} -1 & -1 \\ -4 & \lambda-5 \end{vmatrix} + (-1)^{1+3}\begin{vmatrix} -1 & \lambda \\ -4 & 4 \end{vmatrix}$

$= (\lambda-1)(\lambda(\lambda-5)+4) + 2(-(\lambda-5)-4) + (-4+4\lambda)$

$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$

Thm The roots of the char. poly $p(\lambda) = 0$ are the eigenvalues of the matrix A.

ex Find the eigenvalues and eigenvectors of A in last example.

$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$

$ax^3 + bx^2 + cx + d = 0 \Rightarrow$ formulas

$ax^4 \cdots + e = 0 \Rightarrow$

eigenvalues are roots of $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\downarrow$

$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = 0 \Leftrightarrow \lambda^3 - (\lambda_1+\lambda_2+\lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda$

$-\lambda_1\lambda_2\lambda_3 = 0$

$-\lambda_1\lambda_2\lambda_3 = -6 \Leftrightarrow$ any easy solutions?

integer solutions $\mp 1, \mp 2, \mp 3, \mp 6$.

$\lambda = 1 \Rightarrow 1 - 6 + 11 - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)\left(a\lambda^2 + b\lambda + c\right) = a\lambda^3 + \lambda^2(-a+b) + \lambda(c-b) - c$ $\quad \Rightarrow \begin{array}{l} a = 1 \\ b - a = -6 \Rightarrow b = -6 + a = -5 \\ -c = -6 \Rightarrow c = 6 \end{array}$

$a \to 1$

$p(\lambda) = (\lambda - 1)(\lambda^2 - 5\lambda + 6) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

Eigenvalues: $\lambda = 1, \lambda = 2, \lambda = 3$.

Eigenvectors. $\lambda = 1 \Rightarrow (1 I_3 - A)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 1-1 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & 1-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -4 & 4 & -4 & 0 \end{array}\right] \begin{array}{l} -r_2 + r_3 \\ \to r_3 \end{array}$

$\left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 \leftrightarrow r_2 \\ -r_1 \to r_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2} r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = r \\ x_2 = r/2 \end{array} \quad \begin{array}{l} x_1 - x_2 + x_3 = 0 \\ x_1 - \frac{r}{2} + r = 0 \Rightarrow x_1 = -\frac{r}{2}. \end{array}$

Eigenvectors of $\lambda = 1 \Rightarrow \begin{bmatrix} -r/2 \\ r/2 \\ r \end{bmatrix} = r\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}, r \neq 0$

Another way to write: $s\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, s \neq 0$

Eigenvectors of $\lambda = 2$, $\lambda = 3$ can be found in the same way. Homework. Example 12, 7.1.

ex. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find eigenvalues and eigenvectors.

1st step. Find the char. poly.

$\det(\lambda I_2 - A) = p(\lambda) = \det\begin{pmatrix} \lambda - 0 & +1 \\ -1 & \lambda - 0 \end{pmatrix} = \lambda^2 + 1$

2nd step. $p(\lambda) = \lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1 \Leftrightarrow \lambda_1 = -\underbrace{\sqrt{-1}}_{i}, \lambda_2 = +\underbrace{\sqrt{-1}}_{i}$ $\quad \boxed{i^2 = -1}$

$-i \qquad i$

$\lambda_1 = -i, \lambda_2 = i$.

3rd step. $\lambda_1 = -i \quad \begin{pmatrix} -i - 0 & +1 \\ -1 & -i - 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} -1 & -i & 0 \\ -i & 1 & 0 \end{array}\right] \xrightarrow{-i r_1 + r_2 \to r_2} \left[\begin{array}{cc|c} -1 & -i & 0 \\ -i+i & 1-1 & 0 \end{array}\right]$

$0 \qquad 0$

$-x_1 - i x_2 = 0 \quad \begin{array}{l} x_2 = r \\ x_1 = -ir \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ir \\ r \end{bmatrix} = \underbrace{r\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{\text{eigenvectors.}}, r \neq 0.$

$\lambda_2 = i$ .... Carry on the operations. $\vec{x} = r\begin{bmatrix} i \\ 1 \end{bmatrix}, r \neq 0$

$\hookrightarrow$ can be a complex number.

ex $L: P_2 \to P_2$, $L(at^2 + bt + c) = -bt - 2c$. lin. trans.

(a) Find the matrix representation of L with respect to the basis $S = \{1-t, 1+t, t^2\}$.

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$L(\vec{v}_1) = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$

$L(1-t) = t - 2 = a_1(1-t) + a_2(1+t) + a_3 t^2$ $\quad \begin{array}{l} a_3 = 0 \\ a_2 - a_1 = 1 \\ a_1 + a_2 = -2 \end{array} \quad \begin{array}{l} 2a_2 = -1 \\ a_2 = -1/2 \end{array} \quad a_1 = -3/2$

$\left[L(1-t)\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ 0 \end{bmatrix}$

$\left[L(1+t)\right]_S = \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \end{bmatrix}$

$\left[L(t^2)\right]_S = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The mat. rep. of L w.r.t. ordered basis S is $\begin{bmatrix} -3/2 & -1/2 & 0 \\ -1/2 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$

(b) Find the eigenvalues and eigenvectors of A.

12/05/2020

7.2. Diagonalization.

$A_{2\times2}$ $\quad A\vec{v}_1 = \lambda_1\vec{v}_1 \quad A\vec{v}_2 = \lambda_2\vec{v}_2 \quad \lambda_1 \neq \lambda_2.$ $\quad \vec{v}_1, \vec{v}_2$: eigenvectors.

$P = [\vec{v}_1 \ \vec{v}_2]_{2\times2}$ $\quad AP = A[\vec{v}_1 \ \vec{v}_2] = [A\vec{v}_1 \ A\vec{v}_2]$

proof. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right]$

first col: $\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$

$= [\lambda_1\vec{v}_1 \ \ \lambda_2\vec{v}_2]$ prove!

$= [\vec{v}_1 \ \vec{v}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

If P^{-1} exists $\Rightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ = a diagonal matrix $= D$

$P^{-1}AP = D \Leftrightarrow \underbrace{PP^{-1}}_{I}AP = PD \Leftrightarrow APP^{-1} = PDP^{-1} \Leftrightarrow A = PDP^{-1}$

Why is this useful?

$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PDDP^{-1} = PD^2P^{-1}$

$A^n = PD^nP^{-1}$

HARD EASY

$D^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$

$D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$

DEF $A_{n\times n}$. If P^{-1} exists and D is diagonal mat. and $A = PDP^{-1}$ then A is called **diagonalizable**.

OPTIONAL

Why is computing A^n important?

$e^A \overset{def}{=} \sum_{n=0}^{\infty} \frac{A^n}{n!}$

Calculus 1: $\frac{d}{dx}e^{ax} = ae^{ax}$

Advanced lin. alg. $\frac{d}{dt}e^{At} = Ae^{At}$. This means: $\underbrace{\frac{d}{dt}\vec{x}(t) = A\vec{x}(t), \ \vec{x}(0) = \vec{x}_0}_{\text{INITIAL VALUE PROBLEM}} \Rightarrow \underbrace{\vec{x}(t) = e^{At}\vec{x}_0}_{\text{SOLUTION}}$

When does P^{-1} exist?

It exists if $\vec{v}_1, \vec{v}_2$ are eigenvectors of distinct eigenvalues.

ex $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ Is A diagonalizable, if so find P, D (diagonal matrix) such that $A = PDP^{-1}$.

Previously we found: $\lambda_1 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\quad \lambda_2 = 3, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

Check: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{1\cdot2 - 1\cdot1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\qquad$ H.W. Check it!

Thm (a) If $A_{n\times n}$ matrix. Suppose $\{\vec{v}_1, \dots, \vec{v}_k\}$ are eigenvectors belonging to distinct eigenvalues. Then $\{\vec{v}_1, \dots, \vec{v}_k\}$ are lin. indep.

(b) If A has n distinct eigenvalues then A is diagonalizable.

(c) A has n lin. indep. eigenvectors if and only if then A is diagonalizable.

ex $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Is A diagonalizable?

$p(\lambda)=\det(\lambda I - A)= \det\begin{pmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{pmatrix} = (\lambda-1)^2 =$ char. poly.

$\lambda=1$ is the only eigenvalue.

Find the eigenvectors. $\begin{pmatrix} 1-1 & -1 \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -x_2=0 \\ 0=0 \end{matrix} \Rightarrow \begin{matrix} x_2=0 \\ x_1=r \end{matrix}$, $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ only 1 indep. eigen vector.

NOT DIAGONALIZABLE

Optional. 4.9. Rank of a matrix.

$A=\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \end{bmatrix}$ Number of independent rows = Number of independent cols. $=2=$ rank of a matrix.

$A_{m\times n}$ matrix. $A\vec{x}=\vec{0}$. Sol. set = nullspace of A (subspace of $\mathbb{R}^n$)

Fundamental Theorem of Lin. Algebra: rank of A + $\underbrace{\text{dim. of nullspace of } A}_{\text{nullity}}$ = number of columns of A.

ex $A=\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \end{bmatrix}$ rank=2 $\underset{FTLA}{\Rightarrow}$ nullity=1

$A\vec{x}=\vec{0}$ has 1 lin. indep. sol.

CHAPTER 1

1.4

37. The linear system $AC\mathbf{x} = \mathbf{b}$ is such that A and C are nonsingular with

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Find all 2×2 matrices with real entries of the form

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

such that $A^2 = I_2$.

Chapter 1 Supplementary Exercises

10. Show that the product of two 2×2 skew symmetric matrices is diagonal. Is this true for $n \times n$ skew symmetric matrices with $n > 2$?

Prove that if $\mathrm{Tr}(A^T A) = 0$, then $A = O$.

CHAPTER 2

2.2

11. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x} = 3\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

14. In the following linear system, determine all values of a for which the resulting linear system has

(a) no solution;

(b) a unique solution;

(c) infinitely many solutions:

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

17. Repeat Exercise 14 for the linear system

$$\begin{aligned} x + y &= 3 \\ x + (a^2 - 8)y &= a. \end{aligned}$$

26. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 3y + 3z &= b \\ 5x + 9y - 6z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

In the following linear system, determine all values of a for which the resulting linear system has

(a) no solution;

(b) a unique solution;

(c) infinitely many solutions:

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

2.3

Using elementary matrices.

7. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

17. Which of the following homogeneous systems have a nontrivial solution?

(a) $$\begin{aligned} x + 2y + 3z &= 0 \\ 2y + 2z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

(b) $$\begin{aligned} 2x + y - z &= 0 \\ x - 2y - 3z &= 0 \\ -3x - y + 2z &= 0 \end{aligned}$$

(c) $$\begin{aligned} 3x + y + 3z &= 0 \\ -2x + 2y - 4z &= 0 \\ 2x - 3y + 5z &= 0 \end{aligned}$$

19. Find all value(s) of a for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

exists. What is A^{-1}?

7. Find the inverse of

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

3. Let B be the matrix obtained from A after the row operations $2\mathbf{r}_3 \rightarrow \mathbf{r}_3$, $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$, $4\mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3$, and $-2\mathbf{r}_1 + \mathbf{r}_4 \rightarrow \mathbf{r}_4$ have been performed. If $\det(B) = 2$, find $\det(A)$.

4. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ -2 & 0 & -1 & -1 \\ 3 & 0 & 0 & -1 \end{bmatrix}$$

by using row operations to obtain upper triangular form.

7. Use the adjoint to compute A^{-1} for

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}.$$

8. Solve the linear system $A\mathbf{x} = \mathbf{b}$ by using Cramer's rule, given

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$