Random Questions and Answers

Please send mistakes and comments to taylan.sengul@marmara.edu.tr

2019-08-21

1. Compare Eulerian Jacobian equation

$$\frac{D}{Dt}j(x,t) = j(x,t)\operatorname{div} u(x,t)$$

with continuity equation

$$\frac{D}{Dt}\rho(x,t) = -\rho(x,t)\operatorname{div} u(x,t)$$

Why are they so similar?

answer. ?

2. Newton's Second Law with Einstein's correction?

answer.

$$F = d(mv)/dt$$

with

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the "rest mass".

2019-08-18

3. If CR equations are satisfied at a point, is it true that the function is complex differentiable at that point?

answer. No! See my complex analysis notes.

4. What additional conditions to put so that CR equations imply the complex differentiability?

answer. If in addition, the real and imaginary parts are real differentiable. See my complex analysis notes.

5. How to recall Cauchy-Riemann equations?

answer. The Jacobian matrix is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

Let

$$J = u_x v_y - u_y v_x$$

If CR equations

$$v_y = u_x, \qquad v_x = -u_y$$

are satisfied then the Jacobian determinant

$$J = u_x^2 + u_y^2$$

is always non-zero if $f'(z) = 0$.

6. Prove that L'hopital's rule carries to the complex functions. That is if f and g are two complex-valued functions differentiable at $z = a$ such that $f(a) = g(a) = 0$ and $g'(a) \neq 0$ then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}.$$

<u>`answer`</u>.

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \frac{z - a}{g(z) - g(a)}$$

7. Which of the following are complex differentiable? $\overline{z}$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$

<u>`answer`</u>. None.

8. What are the differences/similarities between a tornado and a hurricane?

<u>`answer`</u>. - Both have cyclonic rotation (although on some rare occasions anti-cyclonic tornadoes may occur)

- Most obvious difference is they have drastically different length scales.

9. What is the typical scales for tornadoes?

<u>`answer`</u>. Most tornadoes have wind speeds less than 180 km/h, are about 80 m across, and travel a few miles (several kilometers) before dissipating. Typical life time is in minutes.

10. What is the time derivative of the deformation gradient? Prove it.

<u>`answer`</u>. $\frac{d\mathbf{F}(\mathbf{a},t)}{dt} = \mathbf{L}(\varphi(\mathbf{a},t),t)\mathbf{F}(\mathbf{a},t)$.

Proof.

$$\begin{aligned}
\frac{d\mathbf{F}}{dt} &= \frac{\partial \mathbf{F}}{\partial t}(\mathbf{a},t) = \nabla_{\mathbf{a}} \frac{\partial \varphi(\mathbf{a},t)}{\partial t} = \nabla_{\mathbf{a}} \mathbf{U}(\mathbf{a},t) = \nabla_{\mathbf{a}} \mathbf{u}(\varphi(\mathbf{a},t),t) \\
&= \left(\frac{\partial u_i}{\partial x_k} \frac{\partial \varphi_k}{\partial a_j} \right) = \mathbf{LF}
\end{aligned}$$

Easier Notation.

$$\dot{\mathbf{F}} = \frac{d}{dt}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}}\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{LF}$$

11. Make sense of

$$\mathbf{J}_{\mathbf{f}^{-1}} \circ \mathbf{f} = \mathbf{J}_{\mathbf{f}}^{-1}$$

answer. According to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function. That is, if the Jacobian of the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and nonsingular at the point p in $\mathbb{R}^n$, then f is invertible when restricted to some neighborhood of p.

Conversely, if the Jacobian determinant is not zero at a point, then the function is locally invertible near this point, that is, there is a neighborhood of this point in which the function is invertible.

2019-08-17

12. A nondegenerate singular point of a smooth vector field is isolated. Why?

answer. By inverse function theorem, locally $u(p) = 0$ is the unique solution since $\det Du(p) \neq 0$.

13. To define the index of a singular point of a smooth vector field, what is the requirement?

answer. The singular point must be isolated. Then any sufficiently small curve around that singular point contains only that isolated point.

14. What is the homotopy invariance property of topological degree?

answer. Let $t \mapsto f_t, t \in [0,1]$ be a continuous path in $C^1(\overline{\Omega})$, with $p \notin f_t(\partial\Omega)$ for all $t \in [0,1]$. Suppose p is a regular value for both f_0 and f_1, then $\deg(f_0, \Omega, p) = \deg(f_1, \Omega, p)$.

2019-08-14

15. Define sectorial operator.

answer. We call a linear operator A in a Banach space X a sectorial operator if

- it is closed and densely defined,
- Location of resolvent: for some $\phi \in (0, \pi/2)$ and real a, the sector

$$S_{a,\phi} = \{\lambda \mid \phi \leq |\lambda - a| \leq \pi, \qquad \lambda \neq a\}$$

is in the resolvent set of A

- Bound on the norm of the resolvent: For some $M \geq 1$

$$\left\|(\lambda - A)^{-1}\right\| \leq \frac{M}{|\lambda - a|} \qquad \forall \lambda \in S_{a,\phi}$$

16. Are linear bounded operators on Banach spaces sectorial?

    `answer`. Yes.

17. If A is densely defined operator in a Hilbert space what is a condition on A so that it is sectorial?

    `answer`. Bounded from below.

18. Define analytic semigroup on a Banach space.

    `answer`. It is a semigroup of bounded linear operators on X such that $t \to T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.

19. Define the infinitesimal generator of a semigroup $\{T(t)\}_{t\geq 0}$. `answer`. $Ax = \lim_{t\to 0+} \frac{T(t)x - x}{t}$ with domain $D(A)$ consisting of all $x \in X$ for which this limit exists. We usually write $T(t) = e^{At}$.

    It is the right derivative of the semigroup at the identity.

20. In a sentence, what is the relation between sectorial operators and analytic semigroups?

    `answer`. If A is a sectorial operator then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$.

21. Formal definition of a power of a sectorial operator?

    `answer`. If A is sectorial and $\operatorname{Re}\sigma(A) > 0$ then for any $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt$$

22. How to define fractional powers of a Banach space which is the domain of a sectorial operator?

    `answer`. $X^\alpha = D(A_1^\alpha)$ with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ where $A_1 = A + aI$ with a chosen so that $\operatorname{Re}\sigma(A_1) > 0$. Different choices of a give equivalent norms.

23. Extend the operator

$$A\phi(x) = -K\frac{d^2\phi}{dx^2}(x), \quad 0 < x < \ell$$

    whenever ϕ is a smooth function on $[0, \ell]$ with $\phi(0) = 0, \phi(\ell) = 0$ to a linear operator on $\mathrm{L}^2(0, \ell)$.

answer. Since A is positive definite

$$(A\phi,\phi) = -K\int_0^\ell \phi''(x)\phi(x)dx = K\int_0^\ell (\phi'(x))^2\,dx \geq 0$$

and symmetric

$$(\mathrm{A}\phi,\psi) = -\mathrm{K}\int_0^\ell \phi''(\mathrm{x})\psi(\mathrm{x})\mathrm{dx} = (\phi,\mathrm{A}\psi)$$

for smooth functions, using Friedrichs theorem, it extends to a self-adjoint, densely defined operator on $L^2(0,\ell)$ with domain

$$D(A) = \left\{\phi \in L^2(0,\ell) | A\phi \in L^2(0,\ell)\right\} = H_0^1(0,\ell)\cap H^2(0,\ell)$$

2019-08-12

24. Chafee and Infante equation? Well-posedness of the IVP? Long time behavior? How many equilibria? Stability of equilibria?

answer.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^3, \quad (0 < x < \pi, t > 0)$$
$$u(0,t) = 0, \quad u(\pi,t) = 0$$

where a, b are positive constants.

IVP is well-posed in $H_0^1(0,\pi)$ and a global solution exists.

Further, as $t \to +\infty, u(\cdot,t)$ converges in $H_0^1(0,\pi)$ to some equillibrium ϕ

$$\frac{d^2\phi}{dx^2}(x) + a\phi(x) - b\phi^3(x) = 0, \quad 0 < x < \pi$$
$$\phi(0) = 0, \quad \phi(\pi) = 0$$

Chafee and Infante prove there are only a finite number of such equilibria - precisely $2n+1$ if $n^2 < a \leq (n+1)^2$ for some integer $n \geq 0$. If $0 < a \leq 1$, the zero solution is thus globally asymptotically stable. If $a > 1$, the zero solution is unstable, as are all other equilibrium points except for two, denoted ϕ_1^+, ϕ_1^-, which both have a dense basin of attraction.

There is a global attractor which is n dimensional if $n^2 < a \leq (n+1)^2$. See pg.5 in Henry.

2019-08-08

25. Define the material derivative of $f(x,t)$ in terms of the motion.

answer. $\frac{D}{Dt}f(x,t) = \frac{\partial}{\partial t}f(\phi(a,t),t)\big|_{\phi(a,t)=x}$.

2019-08-06

26. What is f plane approximation?

    answer. The variation of the Coriolis parameter f w.r.t. the latitude is ignored, a value of f appropriate for a particular latitude is used throughout the domain.

27. Derive the beta plane approximation.

    answer. Taylor approximation of the Coriolis parameter f at a given latitude ϕ_0 is

    $$f = 2\Omega \sin\phi \approx f_0 + \beta y$$

    where y is the meridional distance, Ω is the angular rotation rate of the Earth

    $$f_0 = 2\Omega \sin\phi_0$$

    and

    $$\beta = (df/dy)|_{\phi_0} = 2\Omega \cos(\phi_0)/a$$

    a is the Earth's radius and β is called the Rossby parameter.

    Here we use the fact that $\theta = \frac{y}{a}$ since angle is the arc length over radius.

28. Define convective instability in words.

    answer. Instability arising from cold/dense fluid over warm/light fluid.

29. Define the origin of the word baroclinic.

    answer. baro (pressure) + cline (slope):

    baroclinic instability = instability arising from sloped pressure contours

30. Why is baroclinic instability the most important hydrodynamic instability?

    answer. It is the instability that gives rise to the large-scale and mesoscale motion in the atmosphere and ocean - it produces atmospheric weather systems, for example - and so is, perhaps, the form of hydrodynamic instability that most affects the human condition" G.K. Vallis

2019-08-05

31. What is the vortex stretching term?

    answer. $(\vec{\omega} \cdot \vec{\nabla})\vec{v}$

32. For

    $$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{v}$$

    how does the vortex stretching term effect the dynamics?

    answer. It amplifies the vorticity $\vec{\omega}$ when the velocity is diverging in the direction of $\vec{\omega}$.

33. What is the importance of vortex stretching?

<u>`answer`</u>. Vortex stretching is at the core of the description of the turbulence energy cascade from the large scales to the small scales in turbulence.

2019-08-04

34. Is it possible for a non-holomorphic function to have a primitive?

<u>`answer`</u>. No because the primitive is always holomorphic and the derivative of a holomorphic function is always holomorphic.

35. Is it possible that a holomorphic function does not have a primitive?

<u>`answer`</u>. If the domain is simply-connected it is not possible since a primitive can be constructed by Green's Theorem.

Otherwise it is possible. For example the function 1/z has no primitive on the punctured complex plane as its path integral around the origin on any closed curve is non-zero.

36. Can a discontinuous real valued function have a primitive? Prove or give a counterexample.

<u>`answer`</u>. Yes there are everywhere differentiable functions with discontinuous derivatives at a point such as

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

with derivative

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

which is discontinuous at $x = 0$. The derivative may behave much worse, see (HDCADB).

37. (HDCADB) How discontinuous can the derivative of $f : \mathbb{R} \to \mathbb{R}$ be?

<u>`answer`</u>. There is a well-known result in elementary analysis due to Darboux which says if the derivative of a function satisfies the intermediate value property.

The continuity set is dense and has cardinality c. On the other the discontinuity set can also be dense, have cardinality c and have positive (even full) measure (hence the function can fail to have Riemann integral). `https://math.stackexchange.com/questions/112067/how-discontinuous-can-a-deri`

38. Why is $d\theta$ not exact on the punctured plane?

<u>`answer`</u>. Because θ is not continuous there.

39. If Coriolis force pushes the trajectories to right then why cyclones spin counter clockwise in the northern hemisphere?

    answer. Wind is trying to get into the low pressure center from all directions.

40. Find a basis for $H^1_{dR}(\mathbb{R}^2 \setminus \{0\})$.

    answer. $v = (-y/r^2, x/r^2)$.

41. Write dimension d of $H^2_{dR}(X)$, $X \subset \mathbb{R}^3$ in terms of vector calculus?

    answer.

$$V = \{F : X \to \mathbb{R}^3 : \nabla \times F = 0\}$$
$$W = \{F : X \to \mathbb{R}^3 : F = \nabla g\}$$
$$d = dim(V/W)$$

42. Geometrically the structure of gradient fields vs that of irrotational fields is related to *

    answer. the number of "holes" in the space.

43. What is the aspect ratio (height/length), for surface waves in deep water or convection cells?

    answer. O(1).

44. What is data assimilation?

    answer. Combined observational/numerical modeling simulations

45. At which latitudes there is only water?

    answer. Between 85-90 degrees N and between 55-60 degrees S.

46. At which latitudes there is only land?

    answer. At latitudes 70-90S (Antarctica).

47. What is ocean/earth ratio?

    answer. 0.71

48. What are the x=east, y=north, z=vertical components of the rotation vector in terms of latitude?

    answer.

$$\mathbf{\Omega} = \Omega \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix}$$

49. Which non-zero components (east-west, north-south, up-down) does the Coriolis force have?

    answer. All when the velocity field has all non-zero components.

    See https://en.wikipedia.org/wiki/File:Earth_coordinates.svg

$$\boldsymbol{\Omega} = \Omega \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_e \\ v_n \\ v_u \end{pmatrix}$$

$$\boldsymbol{a}_C = -2\boldsymbol{\Omega} \times \boldsymbol{v} = 2\Omega \begin{pmatrix} v_n \sin\varphi - v_u \cos\varphi \\ -v_e \sin\varphi \\ v_e \cos\varphi \end{pmatrix}$$

    At the equator, $\sin\varphi = 0$ and $\mathbf{a}_C = \mathbf{0}$ if $v_e = v_u = 0$. At the poles, $\mathbf{a}_C = \mathbf{0}$ if $v_e = v_n = 0$.

50. Given

$$\boldsymbol{\Omega} = \Omega \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_e \\ v_n \\ v_u \end{pmatrix}$$

$$\boldsymbol{a}_C = -2\boldsymbol{\Omega} \times \boldsymbol{v} = 2\Omega \begin{pmatrix} v_n \sin\varphi - v_u \cos\varphi \\ -v_e \sin\varphi \\ v_e \cos\varphi \end{pmatrix}$$

    Approximate the Coriolis force in the GFD case.

    answer. In the GFD setting, (1) the vertical velocity v_u is small, and (2) the vertical component $v_e \cos\varphi$ of the Coriolis acceleration is small compared to gravity.

    For such cases, only the horizontal (east and north) components matter. The restriction of the above to the horizontal plane is (setting $v_u = 0$

$$\boldsymbol{v} = \begin{pmatrix} v_e \\ v_n \end{pmatrix}, \quad \boldsymbol{a}_c = \begin{pmatrix} v_n \\ -v_e \end{pmatrix} f$$

    where $f = 2\Omega \sin\varphi$ is called the Coriolis parameter.

51. In the GFD setting, the Coriolis acceleration in the horizontal directions is

$$\boldsymbol{v} = \begin{pmatrix} v_e \\ v_n \end{pmatrix}, \quad \boldsymbol{a}_c = \begin{pmatrix} v_n \\ -v_e \end{pmatrix} f$$

    where $f = 2\Omega \sin\varphi$ is called the Coriolis parameter. Discuss the case of eastward motion and northward motion in the north hemisphere.

    answer. $v_e = 1$ and $v_n = 0$ implies $a_e = 0$ and $a_n = -1$, that is the Coriolis acceleration is due south.

$v_e = 0$ and $v_n = 1$ implies $a_e = 1$ and $a_n = 0$, that is the Coriolis acceleration is due east.

The Coriolis acceleration always points $90°$ to the right of the velocity and is of the same size as the velocity.

52. What is the equatorial speed of earth's rotation?

    answer. 1670 km/hr.

53. Are there currents or winds with comparable speeds to earth's rotation?

    answer. Earth rotates at 1675 km/h. There is no current or wind system on the Earth that approaches speeds of that magnitude. Typical wind speeds in the US are 10-20 km/h. Max wind speed ever measured is 400 km/h.

    For this reason, we describe the winds and currents that we see from within a coordinate frame that removes the basic rotation and shares with us the observational platform of the rotating Earth.

2019-08-01

54. Why is the integral of an arbitrary function of vorticity is conserved for the incompressible barotropic 2D Euler equations?

    answer.

$$\frac{d}{dt}\int_\Omega f(\omega)dV = \int_\Omega \frac{\partial}{\partial t}(f(\omega)) + \nabla\cdot(\mathbf{u}f(\omega))\,dV = \int_\Omega f'(\omega)\frac{D\omega}{Dt}dV = 0$$

    since $\frac{D\omega}{Dt} = 0$.

    For example,

$$\frac{d}{dt}\int_\Omega |\omega|^p\,dV = 0$$

55. List 3 (4) conserved quantity of 3D (2D) Euler's equations.

56. Kinetic energy $\int_{\mathbb{R}^3} |u(x,t)|^2 dx = \int |u_0(x)|^2\,dx$.

    Helicity. $\int_{\mathbb{R}^3} u(x,t)\cdot\omega(x,t)dx = \int_{\mathbb{R}^3} u_0(x)\cdot\omega_0(x)dx$

    Circulation. $\oint_{X(\gamma,t)} u(x,t)\cdot dx = \oint_\gamma u_0(a)\cdot da$

    In 2D, in addition, integrals of arbitrary functions of the vorticity are conserved. In particular the enstrophy $\int_{\mathbb{R}^2} |\omega(x,t)|^2 dx$ is conserved.

57. What is Beale-Kato-Majda criterion?

    answer. For the Euler Equations

$$\int_0^T \|\omega(t)\|_{L^\infty(dx)}dt$$

controls blow up or its absence. If the integral is finite and if the initial velocity is in a Sobolev space H^s with large enough exponent ($s > 5/2$) or in a C^s space with $s > 1$ (and some decay in physical space, for instance $\omega_0 \in L^p$ with $p > 1$) then the solution remains smooth on the time interval $[0, T]$. Of course, if the integral is infinite, then there is finite time blowup.

58. What is the solution to incompressible Euler's equation in 3D?

    <u>`answer`</u>. $\omega(x,t) = (\nabla_a \phi)\,(\phi^{-1}(x,t),t)\omega_0(\phi^{-1}(x,t))$.

    This means that the integral curves of the vorticity, the vortex lines, are carried by the flow.

59. Show that L^p norm of vorticity is controlled for barotropic incompressible 2D Euler equations?

    <u>`answer`</u>. Since $\omega(x,t) = \omega_0(\phi^{-1}(x,t))$ and the "back to labels" map ϕ^{-1} preserves volume it follows that $\|\omega(x,t)\|_{L^p(dx)} = \|\omega_0\|_{L^p(dx)}$ for any $1 \leq p \leq \infty$

60. In 2D, what is the relation between ω, $\nabla^\perp$, $\mathbf{u}$?

    <u>`answer`</u>. $\omega = \nabla^\perp \cdot \mathbf{u}$.

61. Diffusion term in NSE correspond to *

    <u>`answer`</u>. internal processes that lead to energy dissipation.

62. What are nonlocal PDEs?

    <u>`answer`</u>. To check a LOCAL PDE at a particular point, only the values of the function in an arbitrarily small neighborhood are needed, so that all derivatives can be computed. To check a NONLOCAL PDE at a point, information about the values of the function far from that point is needed. Most of the times, this is because the equation involves integral operators.

<u>`2019-07-27`</u>

63. The relation between Helmholtz-Hodge and Biot-Savart. ToDo.

<u>`2019-07-26`</u>

64. Show that for barotropic flows, isobaric (constant pressure) surfaces coincide with constant-density surfaces.

    <u>`answer`</u>. Since density is a function of pressure only, $\rho = \rho(p)$. If p is constant on a surface S so is ρ. If ρ is constant on S and p is not constant on S then $\rho'(p) = 0$ for p values on S. That means ρ is constant for p values on S, a contradiction.

65. Stress tensor for Newtonian fluids?

<u>answer</u>.

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\left(\frac{\partial u_l}{\partial x_l}\right)\delta_{ij}$$

<u>Explanation</u>. The stress tensor in general

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

For Newtonian fluids, the deviatoric part is a linear function of velocity gradient (rate of strain), that is

$$\tau_{ij} = \alpha_{ijkm}\frac{\partial u_k}{\partial x_m}, \qquad i, j, k, m = 1, 2, 3$$

There are a total of $3^4 = 81$ coefficients.

In the isotropic (no preferred directions in the fluid, so the fluid properties are point properties) case and the assumption that no shear stress may act during solid body rotation, this reduces to

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\underbrace{\left(\frac{\partial u_l}{\partial x_l}\right)}_{\nabla\cdot\vec{v}}$$

66. For a Newtonian fluid, the stress tensor is

$$\sigma = -p\mathbf{I} + 2\mu\mathbf{E} + \lambda(\nabla\cdot\mathbf{u})\mathbf{I}$$

where $\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$.

Obtain the momentum equation for an incompressible Newtonian fluid.

<u>answer</u>.

$$\rho\frac{D\mathbf{u}}{Dt} = \nabla\cdot\sigma = -\nabla p + \mu\Delta\mathbf{u}$$

Note that for an incompressible fluid

$$\nabla\cdot(\nabla\mathbf{u})^T = \partial_i(\partial_j u_i) = \partial_j\partial_i u_i = 0$$

67. If a vector field with no singular points exist on a manifold, what can be said about the manifold itself?

<u>answer</u>. The Euler-Poincare characteristic of that manifold is zero.

68. What does a vector field tell about the underlying manifold?

<u>answer</u>. The sum of all the indices of a chosen vector field on a compact differentiable manifold M equals the Euler–Poincare characteristic of M. This sum is ultimately independent of the field which has been chosen — it depends only on the underlying topology of the manifold.

69. Do still objects have vorticity on earth?

answer. Yes they have due to planetary rotation. It is maximum on poles and zero at the equator.

2019-07-25

70. What length scales do turbulence operate?

answer. Turbulence operates on all scales down to millimeters.

71. Why on smaller scales, GFD turns into classical fluid dynamics?

answer. Because effects of planetary rotation and vertical stratification weaken.

72. What are the characteristics of tropical cyclones?

answer. Tropical cyclones (hurricanes and typhoons) are a coupled ocean-atmosphere phenomenon. These are powerful storm systems characterized by low-pressure center, strong winds, heavy rain, and numerous thunderstorms.

73. What do ocean and atmosphere exchange?

answer. Momentum, heat, water, radiation, aerosols, and greenhouse gases.

74. Derive the time derivative of the Jacobian in Eulerian coordinates using the formula for Lagrangian coordinates.

answer.

$$\frac{Dj(x,t)}{Dt} = j(x,t)\left(\nabla_x \cdot u(x,t)\right)$$

where $j(\phi(a,t),t) = J(a,t)$ and $u(\phi(a,t),t) = U(a,t)$.

proof. Combining

$$\frac{\partial J(a,t)}{\partial t} = \frac{D}{Dt} j(\phi(a,t),t)$$

.

$$\nabla_a \cdot U(a,t) = \nabla_x \cdot u(\phi(a,t),t)$$

$$\frac{\partial J(a,t)}{\partial t} = J(a,t)\nabla_a \cdot U(a,t)$$

gives

$$\frac{D}{Dt} j(\phi(a,t),t) = j(\phi(a,t),t)\left(\nabla_x \cdot u(\phi(a,t),t)\right)$$

75. Derive the time derivative of the Jacobian in Lagrangian coordinates.

<u>`answer`</u>.
$$\frac{\partial J(a,t)}{\partial t} = J(a,t)\nabla_a \cdot U(a,t)$$

<u>proof</u>. Define the velocity gradient $L(x,t) = \nabla_x u(x,t)$ (an Eulerian quantity).

$$\begin{aligned}\frac{\partial F(a,t)}{\partial t} &= \frac{\partial}{\partial t}\nabla_a \phi(a,t) = \nabla_a U(a,t) = \nabla_a u(\phi(a,t),t) \\ &= L(\phi(a,t),t)F(a,t)\end{aligned}$$

$$\begin{aligned}\frac{\partial J(a,t)}{\partial t} &= J(a,t)\,\mathrm{tr}\left(\frac{\partial F(a,t)}{\partial t}F^{-1}(a,t)\right) = J(a,t)\,\mathrm{tr}\left(L(x,t)\right) \\ &= J(a,t)\nabla_x \cdot u(x,t)\end{aligned}$$

<u>`2019-07-24`</u>

76. Shallow water approximation.

<u>`answer`</u>. ToDO.

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{U}{L}} + \underbrace{\frac{\partial v}{\partial y}}_{\frac{U}{L}} + \underbrace{\frac{\partial w}{\partial z}}_{\frac{W}{D}} = 0$$

Assume
$$\frac{D}{L} \ll 1$$

We have
$$W = \frac{DU}{L} \ll U$$

Horizontal momentum conservation reads

$$\underbrace{\frac{\partial u}{\partial t}}_{\frac{U}{T}} + \underbrace{\left(u\frac{\partial u}{\partial x}\right.}_{\frac{U^2}{L}} + \underbrace{v\frac{\partial u}{\partial y}}_{\frac{U^2}{L}} + \underbrace{w\frac{\partial u}{\partial z}}_{\frac{U^2}{L}}) - \underbrace{fv}_{fU} = \underbrace{\frac{1}{\rho}\frac{\partial p_d}{\partial x}}_{\frac{P}{\rho L}}$$

where p_d stands for the dynamic pressure

$$p = -\rho g z + p_d$$

The scale of pressure is

$$P = \frac{\rho U L}{T}, \qquad \text{or} \qquad \rho U f L$$

Vertical momentum balance is

$$\underbrace{\frac{\partial w}{\partial t}}_{\frac{W}{T}} + \Big(\underbrace{u\frac{\partial w}{\partial x}}_{\frac{UW}{L}} + \underbrace{v\frac{\partial w}{\partial y}}_{\frac{UW}{L}} + \underbrace{w\frac{\partial w}{\partial z}}_{L}) = \underbrace{\frac{1}{\rho}\frac{\partial p_d}{\partial z}}_{\frac{P}{\rho D}}$$

Then either

$$\frac{\frac{\partial w}{\partial t}}{\frac{1}{\rho}\frac{\partial p_d}{\partial z}} \sim \frac{\frac{DU}{LT}}{\frac{\rho UL/T}{\rho D}} \sim \frac{D^2}{L^2}$$

or

$$\frac{\frac{\partial w}{\partial t}}{\frac{1}{\rho}\frac{\partial p_d}{\partial z}} \sim \frac{\frac{DU}{LT}}{\frac{UfL}{\rho D}} \sim \frac{D^2}{L^2}\frac{1}{Tf}$$

since the time scale of interest is of a day or so, $fT = O(1)$. We conclude that the vertical pressure gradient is practically zero with error of order $D^2/L^2 \ll 1$, implying that

$$p_d \simeq \rho g \eta$$

or the total pressure is hydrostatic

$$p_{\text{ total }} \simeq \rho g(\eta - z)$$

Thus

$$\frac{\partial p_d}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \quad \frac{\partial p_d}{\partial y} = \rho g \frac{\partial \eta}{\partial y}$$

We now show by a formal perturbation scheme for horizontal bottom, that u, v are independent of z to the leading order or approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

77. Compare the trajectories, fixed points and the type of fixed points for

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

and

$$\dot{x} = Q(x, y)$$
$$\dot{y} = -P(x, y)$$

<u>`answer`</u>. The trajectories are orthogonal since the vector fields are orthogonal since $(P, Q) \cdot (Q, -P) = 0$. The critical points are the same. Centers of one correspond to nodes for the other, saddles of one correspond to saddles of the other, and foci correspond to foci.

78. (1)

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

(2)

$$\dot{x} = Q(x, y)$$
$$\dot{y} = -P(x, y)$$

Show that if one of them is a Hamiltonian system, then the other is a gradient system and vice versa.

answer. For instance, suppose (1) is Hamiltonian, then there exists $H(x, y)$ such that $H_y = P$ and $H_x = -Q$. Then (2) is a gradient system with $V(x, y) = H(x, y)$

79. If the first de Rham cohomology vanishes, is the space simply connected?

answer. Yes if $\Omega \subset \mathbb{R}^n$ with $n = 2$. In general no. See https://www.csun.edu/~vcmth02i/Forms.pdf.

80. If $h : X \to Y$ with $h(x_0) = y_0$ is a homeomorphism then prove that $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

answer. A continuous map $h : X \to Y$ with $h(x_0) = y_0$ defines an induced map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([c]) = [h \circ c]$ for any loop c in X centered at x_0.

- The map is well defined. If c and c' are homotopic paths, i.e. $H(t, 0) = c(t)$ and $H(t, 1) = c'(t)$ then $h \circ H(t, 0) = h \circ c(t)$ and $h \circ H(t, 1) = h \circ c'(t)$, i.e. $h_*[c] = h_*[c']$.
- This induced map is a homomorphism since $h_*[c]^{-1} = (h_*[c])^{-1}$ where c^{-1} is the loop at x_0 traversing inversely and $h_*([c_1][c_2]) = h_*[c_1]h_*[c_2]$.
- h_* is injective since $h_*[c_1] = h_*[c_2]$ implies $(h^{-1})_*h_*[c_1] = (h^{-1})_*h_*[c_2]$ which implies $[c_1] = [c_2]$.
- h_* is surjective since $[d] = h_*\left((h^{-1})_*[d]\right)$.

81. Show that torus is not homeomorphic to $\mathbb{R}^2$.

answer. One is simply connected and the other is not.

82. If two spaces are homeomorphic then they have isomorphic fundamental groups. Is the converse true?

answer. No. Take $\mathbb{R}^2$ and $\mathbb{R}^3$ for example.

83. At a critical rotation period T_c, centrifugal force and gravitational force cancels each other at the equator. What does T_c depend on?

answer. It depends on the density as $\frac{1}{\sqrt{\rho}}$ of the object.

$\frac{GM}{R^2} = \Omega^2 R$. Letting $T = \frac{2\pi}{\Omega}$ gives $T_c = 2\pi \left(\frac{R^3}{GM}\right)^{1/2} = \left(\frac{3\pi}{G\rho}\right)^{1/2}$. If $T < T_c$ then the object will be torn apart.

84. What is the critical rotation period for Earth, Jupiter and sun where centrifugal and gravitational force cancels each other?

    answer. For Earth, T_c is 1.4 hours, for Jupiter it is 2.8 hours (actual period is 10 hours), for sun it is also 2.8 hours (sun and Jupiter have approximately equal densities).

    Since earth has almost 4 times the density of sun (or Jupiter) $T_e \rho_e^{1/2} = T_j \rho_j^{1/2}$ so that $2T_e = T_j$.

85. Densities of solar system bodies?

    answer. Earth is densest planet in the solar system. In kg/m3 densities are Sun 1.4, Mercury 5.4, Venus 5.2, Earth 5.5, Mars 3.9, Jupiter 1.3, Saturn 0.7, Uranus 1.3, Neptune 1.6

86. Write $\mathbf{u}$ in terms of motion ϕ? Is $\mathbf{u}(x,t) = \frac{\partial \phi^{-1}(x,t)}{\partial t}$?

    answer. No! $\mathbf{u}(x,t) = \mathbf{U}(\phi^{-1}(x,t),t) = \left.\frac{\partial \phi(a,t)}{\partial t}\right|_{a=\phi^{-1}(x,t)}$.

87. What is the material derivative in Lagrangian coordinates?

    answer. Let $f(\phi(a,t),t) = F(a,t)$. Then

$$\frac{\partial F(a,t)}{\partial t} = \left.\frac{Df(x,t)}{Dt}\right|_{x=\phi(a,t)}.$$

    Thus if a property does not change in time following the fluid parcel (such as density), i.e. $F(a,t) = F(a)$ then $Df/Dt = 0$.

2019-07-23

88. When is 3D baroclinic instability discovered and by who?

    answer. Jule G. Charney (1947) and, independently, by Eric T. Eady (1949).

89. What is the reason for parallel cloud streaks in the sky?

    answer. Either Rayleigh-Bénard or Kelvin-Helmholtz instability.

90. In which latitudes is the planetary rotation more important?

    answer. In higher latitudes.

91. What is the derivative of a map between two manifolds, $f : M \to N$ at a point M? Use derivation definition.

<u>`answer`</u>. Let X be a vector field on M and $g : N \to \mathbb{R}$. Then define

$$df(X)(g) = X(f^*g) = X(g \circ f)$$

92. What is the derivative of a map between two manifolds, $f : M \to N$ at a point M? Use curve definition of tangent vector.

<u>`answer`</u>. It is the linear map $df : TM \to TN$ constructed such that $df(\frac{d\gamma}{dt}(0)) = \frac{d}{dt}(f \circ \gamma)(0)$ where γ is a curve $\gamma(0) = m$ and $\frac{d\gamma}{dt}(0)$ is its tangent vector at m.

<u>`2019-07-22`</u>

93. How can Morse Lemma be used to analyze the stability of non-degenerate equilibria of conservative systems?

<u>`answer`</u>. Dynamics of conservative systems take place on level sets of conserved quantity H. Local level sets of H can be analyzed by Morse Lemma. If the quadratic part has full or zero index then level sets are closed and the equilibrium is a center. Otherwise the equilibrium is a saddle.

94. Taylor Theorem and Morse Lemma?

<u>`answer`</u>. Near a non-degenerate critical point of a function $f : M \to \mathbb{R}$, the behavior of the function is determined by the quadratic part of the Taylor expansion.

95. Describe Cauchy's stress theorem in words.

<u>`answer`</u>. The state of stress at a point in the body is then defined by all the stress vectors associated with all planes (infinite in number) that pass through that point. However, according to Cauchy's fundamental theorem,[11] also called Cauchy's stress theorem, merely by knowing the stress vectors on three mutually perpendicular planes, the stress vector on any other plane passing through that point can be found through coordinate transformation equations.

Cauchy's stress theorem states that there exists a second-order tensor field $\boldsymbol{\sigma}(\mathbf{x}, t)$, called the Cauchy stress tensor, independent of $\mathbf{n}$, such that T is a linear function of $\mathbf{n}$:

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i$$

96. Show that the stress is linear in the normal. This is called Cauchy stress principle.

answer. Consider a small tetrahedral parcel of fluid with three faces normal to the coordinate axes and the fourth face with normal **n**. Across each face, the fluid outside the tetrahedron exerts a force, or traction, on the fluid inside; call the forces $A_x\boldsymbol{T}_x, A_y\boldsymbol{T}_y, A_z\boldsymbol{T}_z$, and A_n respectively where A_i is the area of the i th face and $\boldsymbol{T}_i$ are the stresses.

$$\rho V\frac{d\boldsymbol{q}}{dt} = A_x\boldsymbol{T}_x + A_y\boldsymbol{T}_y + A_z\boldsymbol{T}_z + A_n\boldsymbol{T}_n$$

where V is the volume of the tetrahedron. Hence, crudely, the acceleration must be proportional to (surface area)/(volume) which becomes infinite for arbitrarily small volumes. Such infinite acceleration cannot be permitted, and hence there cannot be any net imbalance of the forces on the tetrahedron. Thus

$$\begin{aligned}\boldsymbol{T}_n &= -\frac{A_x}{A_n}\boldsymbol{T}_x - \frac{A_y}{A_n}\boldsymbol{T}_y - \frac{A_y}{A_n}\boldsymbol{T}_z \\ &\text{by projection of areas} \\ &= (\hat{\boldsymbol{n}}\cdot\boldsymbol{i})\boldsymbol{T}_x + (\hat{\boldsymbol{n}}\cdot\boldsymbol{j})\boldsymbol{T}_y + (\hat{\boldsymbol{n}}\cdot\boldsymbol{k})\boldsymbol{T}_z\end{aligned}$$

From this we see that the stress across a plane with any normal $\hat{\boldsymbol{n}}$ is linear in $\hat{\boldsymbol{n}}$: for some stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{T}_n = \hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}$$

Indeed, in a matrix representation $\boldsymbol{T}_x, \boldsymbol{T}_y$ and $\boldsymbol{T}_z$ form the "rows" of the stress tensor:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

where σ_{ij} is the component of the stress in the j th coordinate direction upon a plane with normal in the direction of the i th coordinate.

97. Consider a small tetrahedral parcel of fluid with three faces normal to the coordinate axes and the fourth face with normal **n**. Let the surface areas be A_x, A_y, A_z and A_n. What is $\frac{A_x}{A_n}$?

answer. $-\hat{\mathbf{n}}\cdot\mathbf{i}$.

2019-07-21

98. (A) When does $f : D \subset \mathbb{C} \to \mathbb{C}$ have a primitive? (B) PROVE it! (C) Is primitive unique?

answer. (A) If D is simply connected and f is holomorphic on D (both conditions are necessary, think $f(z) = 1/z$ for simply-connectedness and the fact that derivative of a holomorphic function (the primitive) is holomorphic). (B) In this case define

$$F(z) = \oint_{z_0}^{z} f(\zeta) d\zeta, \quad z \in D$$

where $z_0 \in D$ is fixed point and the integral can be taken along any path in D from z_0 to z. By Cauchy's integral theorem, the integral does not depend on the curve but only on the endpoints. Hence F is well-defined.

To show that $F'(z) = f(z)$, let $h > 0$. Then

$$\left| \frac{F(z+h) - F(z)}{h} \right| \leq \int_{z}^{z+h} |f(\zeta)|\, d\zeta \leq \sup_{\zeta \in [z, z+h]} |f(\zeta)|\, |h| \to 0$$

as $h \to 0$ by continuity. Here $[z, z+h]$ is the line segment. Similar argument holds for $h < 0$.

(C) The solution is unique up to a constant. (The choice of z_0).

99. If $f(z)$ is continuous on a domain D, and if $F(z)$ is a primitive for $f(z)$,

$$\int_{A}^{B} f(z) dz = F(B) - F(A)$$

where the integral can be taken over any path in D from A to B. Prove it!

answer.

$$\int_{\gamma(t)} f(z) dz = \int_{0}^{1} f(\gamma(t)) \gamma'(t) dt = \int_{0}^{1} \frac{d}{dt} F(\gamma(t)) dt$$

2nd way.

$$F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i} \frac{\partial F}{\partial y}$$

so that

$$\begin{aligned} F(B) - F(A) &= \int_{A}^{B} dF = \int_{A}^{B} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \\ &= \int_{A}^{B} F'(z)(dx + i dy) = \int_{A}^{B} f(z) dz \end{aligned}$$

100. The proof of Green's theorem is very interesting. See Gamelin pg. 90. ToDO.

101. Pullback the the line integral $\int_{S^1} xy dy$ to $[0, 2\pi]$

answer. $\int_0^{2\pi} \cos t \sin t \cos t dt$.

102. Intuitively what can be said about the isolated local minima/maxima of V for the gradient system $\dot{x} = -\operatorname{grad} V(x)$?

<u>`answer`</u>. Isolated local minima are locally asymptotically stable, isolated local maxima are unstable. The system is pushing toward where V is most decreasing, seeking the minima and getting away from the maxima of V.

103. Show that the isolated local minima of V for the gradient system $\dot{x} = -\operatorname{grad} V(x)$ are locally asymptotically stable.

<u>`answer`</u>. Suppose $V(a)$ is a local isolated minimum. Then $L(x) = V(x) - V(a)$ is a local strict Lyapunov function.

- $\frac{d}{dt}(V(x(t)) = \operatorname{grad} V(x)\dot{x} < 0$ (since $\dot{x} \neq 0$), for $x \neq a$ (locally)
- $L(x) > 0$ (locally)
- $L(a) = 0$

104. Show that the isolated local maxima of V for the gradient system $\dot{x} = -\operatorname{grad} V(x)$ are unstable using the fact that local isolated minima are locally asymptotically stable.

<u>`answer`</u>. Do time reversal $s = -t$. Then $x' = -\operatorname{grad} W(x)$ where $W(x) = -V(x)$. If $V(a)$ is a local isolated maxima then $W(a)$ is local isolated minima.

105. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) =?$

<u>`answer`</u>. $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

<u>`2019-07-20`</u>

106. What is the index of a non-degenerate singular point of a vector field?

<u>`answer`</u>. ± 1.

107. Why is the index of a nondegenerate singular point of vector field u ± 1?

<u>`answer`</u>. Let $D = det(Du(p))$. Since u is nondegenerate at p, $D \neq 0$. By inverse function theorem, locally, p is the unique solution to $u(x) = 0$. Hence by the topological degree, index of p is $sign(D)$ which is ± 1.

<u>`2019-07-19`</u>

108. What is inertial period? ToDO.

109. What is the range of numerical values of Coriolis parameter?

<u>`answer`</u>.

$$1.4 \times 10^{-4} s^{-1} \leq 2\Omega \sin\phi \leq 0 s^{-1}$$

Here $\Omega \approx 7 \times 10^{-5} s^{-1}$ and ϕ is the latitude (1 at the poles, 0 at the equator).

2019-07-18

110. Show that the Euler characteristic of the torus is 0 by the Morse theory.

`answer`. Take the z-height function on the torus. There are four non-degenerate critical points, a max with index 2, a min with index 0 and two saddles with index 1.

$$\chi(N) = \sum_{i=0}^{n} (-)^k n_k = (-1)^2 + (-1)^1 \times 2 + (-1)^0 = 0$$

where n_k denotes the number of critical points with index k.

111. How does effective gravity work?

`answer`. The true gravitational acceleration (g*) pulls an object towards the center of mass of the earth. However, the centrifugal force pushes all objects outward from the axis of planetary rotation. The effective gravity (g) is the vector sum of these two forces. It does not point directly at the center of earth mass.

112. What is the relative strength of centrifugal force to gravitational force on earth?

`answer`. Within the earth's atmosphere the magnitude of the centrifugal force is less than 0.03 percent of g.

113. Does Rossby number control whether relative vorticity vs planetary vorticity dominates?

`answer`. I think this. ToDo

114. For a fast current such as Gulf stream, what is the order of magnitude comparison of relative vs planetary vorticity?

`answer`. $\zeta \ll f$, where ζ is the relative, f is the planetary vorticity. ζ is usually much smaller than f, and it is greatest at the edge of fast currents such as the Gulf Stream. To obtain some understanding of the size of ζ, consider the edge of the Gulf Stream off Cape Hatteras where the velocity decreases by 1 m/s in 100 km at the boundary. The curl of the current is approximately (1m/s)/(100 km) = 0.14 cycles/day = 1 cycle/week. Hence even this large relative vorticity is still almost seven times smaller than f. A more typical values of relative vorticity, such as the vorticity of eddies, is a cycle per month.

115. In 2D, classify the linear stability of non-degenerate equilibria of divergence free (for example Hamiltonian) systems.

`answer`. Nondegenerate equilibria can only be saddle or center. Degenerate equilibria is harder to describe.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\dot{x} = f(x)$ with $\nabla \cdot f(x) = 0$. Then

$$\operatorname{tr}(Df(x)) = \nabla \cdot f(x) = 0, \qquad d = \det(Df(x)) \neq 0$$

by non-degeneracy. Thus eigenvalues are $\pm\sqrt{d}$ which correspond to **a saddle** if $d > 0$ and **a center** if $d < 0$.

2019-07-16

116. Cauchy's integral formula $\int_C ? f(z) dz = f(z_0)$

answer. $\dfrac{1}{2\pi i(z - z_0)}$

Here f is complex-differentiable in an open region $U \subset \mathbb{C}$ which contains a simple counterclockwise loop C and the region bounded by $C$and z_0 is arbitrary in the interior of the region bounded by C.

117. What is the intution behind

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0)$$

answer.

$$\int_C \frac{f(z)}{z - z_0} dz = \int_C \frac{1}{z - z_0}(f(z_0) + f'(z_0)(z - z_0) + \cdots) = f(z_0) \int_C \frac{1}{z - z_0}$$

The last integral is $2\pi i$ for unit circle. And by homotopy, for any curve homotopic to unit circle.

118. Define cross product using determinant.

answer. $\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} u_1 & v_1 & \boldsymbol{e}_1 \\ u_2 & v_2 & \boldsymbol{e}_2 \\ u_3 & v_3 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 & - & u_3 v_2 \\ u_3 v_1 & - & u_1 v_3 \\ u_1 v_2 & - & u_2 v_1 \end{bmatrix}$

119. Can isolated critical points be degenerate?

answer. Yes. Take $f(x, y) = x^2 + y^4$.

120. What are the possibilities of the character of non-degenerate equilibrium points of a conservative system?

answer. By Morse Theorem, they can be either centers (Morse index full or zero), or saddles (Morse index between full and zero).

121. Can non-degenerate critical points of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ be non-isolated? Why?

answer. No. By Morse Lemma, near a non-degenerate critical point, the function behaves like a quadratic polynomial whose gradient field is non-vanishing except at the singular point.

122. Can non-degenerate singular points $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ of a vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be non-isolated? Why?

answer. No. By the inverse function theorem, near $\mathbf{p}$, $\mathbf{f}$ is locally invertible.

123. Suppose a first integral E of an ODE has an nondegenerate critical point x_0. (1) Show that x_0 is an equilibrium point. (2) What can be said about the character of the fixed point?

answer. (1) Note that solutions of ODE lie on the level sets of E. Since the singleton $\{x_0\}$ is a level set, it must be an equilibrium point.

(2) x_0 is a center of the ODE if it is a strict minimum/maximum of E and it is a saddle of the ODE if it is a saddle of E.

To see, in a small neighborhood of x_0 there is a coordinate transformation which takes E to a quadratic function. If x_0 is an isolated min or max then level surfaces are spheres. If x_0 is an isolated saddle of E then it is a saddle of the ODE.

What if x_0 is a degenerate isolated point? We can not use Morse Lemma. ToDO.

2019-07-15

124. How to define index of a vector field on a manifold?

answer. ToDO.

125. Compute $\oint_\gamma z^n dz$ where γ is the unit circle.

answer. Put $z = e^{i\theta}$. Then

$$\oint_\gamma z^n dz = \int_0^{2\pi} e^{in\theta} e^{i\theta} i d\theta = \int_0^{2\pi} e^{i(n+1)\theta} d\theta = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}$$

126. State $\oint_\gamma z^n dz$ for a simple closed loop γ around the origin and $n \in \mathbb{Z}$.

answer.

$$\begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}$$

127. Under which conditions $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$?

answer. When C_i are homotopic paths in a region D and $f : D \to \mathbb{C}$ is holomorphic.

128. When C_i are homotopic paths with fixed endpoints in a region D and $f : D \to \mathbb{C}$ is holomorphic, PROVE that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

<u>`answer`</u>. Let $H(t, s)$ be a smooth homotopy between smooth C_1 and C_2, fixing the For each s, the function $\gamma_s(t)$ describes a closed curve C_s in D. Let $I(s)$ be given by

$$I(s) = \int_{C_s} f(z)dz = \int_0^1 f(H(t,s))\frac{\partial H(t,s)}{\partial t}dt$$

Using

$$\frac{\partial}{\partial s}\left[f(H(t,s))\frac{\partial H(t,s)}{\partial s}\right] = \frac{\partial}{\partial t}\left[f(H(t,s))\frac{\partial H(t,s)}{\partial s}\right]$$

$$\begin{aligned}\frac{d}{ds}I(s) &= \int_0^1 \frac{\partial}{\partial t}\left[f(H(t,s))\frac{\partial H(t,s)}{\partial s}\right] dt \\ &= 0\end{aligned}$$

since $\frac{\partial H(1,s)}{\partial s} = \frac{\partial H(0,s)}{\partial s} = 0$ because the endpoints of the homotopy are fixed.

Smoothness assumption can be dropped by approximation.

129. A good mathematics trick.

$$\frac{\partial}{\partial s}\left(f(H(t,s))\frac{\partial H(t,s)}{\partial t}\right) = \frac{\partial}{\partial t}\left(f(H(t,s))\frac{\partial H(t,s)}{\partial s}\right)$$

How to generalize and recall this?

<u>`answer`</u>. $F' = f$ exists since f is differentiable (hence continuous). Then LHS is $\frac{\partial}{\partial s}\frac{\partial}{\partial t}F(H(t,s))$ and RHS is $\frac{\partial}{\partial t}\frac{\partial}{\partial s}F(H(t,s))$.

130. Which real valued functions have a primitive?

<u>`answer`</u>. A sufficient but not necessary condition is that continuous functions have primitives.

Discontinuous ones can also have anti derivative. Wikipedia's page on anti derivative says there are open problems in this question.

A necessary but not sufficient condition is that the function should have the intermediate value property.

131. State the Cauchy's integral theorem.

<u>`answer`</u>. If $f : D \to \mathbb{C}$ is holomorphic and the open set D is simply-connected then

$$\int_\gamma f(z)dz = 0$$

for any rectifiable (having finite length) closed path in D.

132. Prove the Cauchy's integral theorem. If $f : D \to \mathbb{C}$ is holomorphic and the open set D is simply-connected then

$$\int_\gamma f(z)dz = 0$$

for any rectifiable (having finite length) closed path in D.

answer.

$$f(z)dz = (u + iv)(dx + idy) = (udx - vdy) + i(vdx + udy)$$

Both real forms are closed, thanks to Cauchy-Riemann equations. On a simply connected domain, closed 1-forms are exact. Hence their line integrals on closed paths are always zero. The result follows.

133. What is the sign of Green's function to Laplace operator in $\mathbb{R}^3$? Why?

answer. $G = -\dfrac{1}{4\pi r}$. Reason: ∇G must have positive outward flux on any sphere around origin. For $G = -\dfrac{1}{4\pi r}$, ∇G points outward as G is radially increasing from origin. $\int_{B_r(0)} \nabla \cdot \nabla G dV = \int_{S_r(0)} \nabla G \cdot \widehat{\mathbf{r}} dS > 0$.

2019-07-14

134. Define cyclonic rotation.

answer. Counter clockwise in the Northern Hemisphere and clockwise in the Southern Hemisphere

135. In a Hamiltonian system, what is a necessary requirement on the spectrum of the linearized operator for the nonlinear stability of equilibrium?

answer. The spectrum must lie entirely on the imaginary axis. Otherwise since eigenvalues come in pairs $\pm\lambda$, the equilibrium is always unstable.

136. What are the eigenvalues of an Hamiltonian matrix?

answer. If $\lambda \in \mathbb{C}$ is an eigenvalue so are $\pm\lambda$, $\pm\overline{\lambda}$.

Thus eigenvalues occur in the following configurations: (a) imaginary pairs $\pm i\omega, \omega \in \mathbb{R}^+$, (b) real pairs $\pm\sigma, \sigma \in \mathbb{R}^+$, (c) complex quadruplets $\pm a \pm ib, a, b \in \mathbb{R}^+$, (d) $\sigma = 0$.

Moreover, $\pm\sigma$, and the complex conjugates $\pm\sigma^*$ all have the same multiplicity and Jordan block structure, while a zero eigenvalue has even multiplicity.

To see, note that characteristic polynomial of a Hamiltonian matrix is real and even. Thus if $p(\lambda) = 0$ then $p(-\lambda) = 0$ and $\overline{p(\lambda)} = p(\overline{\lambda}) = 0$.

137. What is the property of the characteristic polynomial of a Hamiltonian matrix?

answer. It is an even polynomial, $p_A(-x) = p_A(x)$.

138. Suppose A is Hamiltonian, i.e. $A = JA^TJ$ where $J^2 = -I$. Show that its characteristic polynomial is even.

answer.

$$p_A(x) = \det(A - xI) = \det\left(JA^TJ + xJIJ\right) = (\det J)^2 \det(A^T + xI) = p_A(-x)$$

since $(\det J)^2 = \det(-I_{2n}) = 1$ and $\det(A^T + xI) = \det(A^T - (-x)I) = p_{A^T}(-x) = p_A(-x)$.

139. Hamiltonian matrices and Lie algebra/group structure?

answer. Hamiltonian matrices (matrices of the form JS with S symmetric and J nonsingular skew-symmetric) are closed under addition, scalar product and matrix commutator hence form a Lie algebra. The corresponding Lie group is the symplectic group $Sp(2n, F)$ of $2n \times 2n$ symplectic matrices.

140. What is a Hamiltonian matrix and where does it arise?

answer. A Hamiltonian matrix is a matrix of the form JS where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and S is symmetric.

They arise in the linearizations of the Hamiltonian systems. If

$$\dot{z} = JDH$$

then the linearized system around $z = z_0$ is

$$\dot{y} = JD^2H(z_0)y$$

141. Find $ADf(x)$ where $A : \mathbb{R}^n \to \mathbb{R}^m$ is constant and linear and $f : \mathbb{R}^k \to \mathbb{R}^n$.

answer. $ADf(x) = DAf(x)$. To see,

$$[ADf(x)]_{ik} = a_{ij}f_{j,k} = \partial_k(a_{ij}f_j) = [DAf(x)]_{ik}$$

since $(Df)_{jk} = f_{j,k} = \partial_k f_j$.

142. If $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear, what is the size of the matrix?

answer. $A_{m\times n}x_{n\times 1} = b_{m\times 1}$.

143. If $f : \mathbb{R}^k \to \mathbb{R}^n$, what is the size of the matrix $Df(x)$?

answer. i-th row is the derivative of f_i, $1 \le i \le n$. So there are n rows. $Df(x)$ is $n \times k$.

144. What does the spectrum of the linear operator imply about the the linear/nonlinear stability around an equilibrium point q in ODEs?

 answer. LS/LU = equilibrium is linearly stable/unstable

 NS/NU = equilibrium is nonlinearly stable/unstable

 If $\Re\lambda_i > 0$ for some i then LU-NU

 If $\Re\lambda_i < 0$ for all i then LS-NS

 If $\Re\lambda_i \leq 0$ for all i then LS-NS, LS-NU, LU-NS, LU-NU are all possible.

 See (LS-NSEx) for examples.

145. (LS-NSEx) For an ODE, let

 - LS/LU = equilibrium is linearly stable/unstable
 - NS/NU = equilibrium is nonlinearly stable/unstable

 Give examples of all combinations (if they exist).

 answer. LS-NS $\dot{x} = -x - x^3$, for LS-NU $\dot{x} = x + x^3$, for LU-NU $\dot{x} = x$. For LU-NS, take the Hamiltonian $H(q,p) = \frac{p^2}{2} + \frac{q^4}{4}$. Since $(0,0)$ is an isolated minimum, the origin is a nonlinear center. The linearization at the equilibrium is $\dot{q} = p$, $\dot{p} = 0$, i.e. $\ddot{q} = 0$, the solution grows linearly in time.

146. For a conservative system, show that linear instability of a fixed point does not imply instability.

 answer. This is true for a Hamiltonian system hence for a conservative system. See (HamExLU-NS).

147. (HamExLU-NS) For a Hamiltonian system, show that linear instability of a fixed point does not imply instability.

 answer. Take the Hamiltonian $H = p^4 + q^2$. The origin is an equilibrium point. The linearized equations are $dq/dt = 0, dp/dt = -2q$. This has solution $q = q_0$, $p = -2q_0t + p_0$ which is increasing linearly with time, so the origin is unstable for the linearized system. However, the Hamiltonian has a minimum there, so it is stable.

148. For degenerate fixed points of a conservative system, linear instability of a fixed point does not imply instability, see (HamExLU-NS). What about non-degenerate fixed points?

 answer. A non-degenerate fixed point of a conservative system can only be a center or a saddle. Suppose $E = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2$.

149. Linearized equations of a Hamiltonian system is a Hamiltonian system? True or False?

 answer. ?

150. How to recall the formula for the divergence of cross product?

answer. $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})$.

The result is a scalar, skew-symmetric, includes only 1st order derivative operator.

151. Write the effective gravity formula.

answer. $\boldsymbol{g} \equiv \boldsymbol{g}_{\text{eff}} = \boldsymbol{g}_{\text{grav}} + \boldsymbol{g}_{\text{cf}} = g\mathbf{k} + \Omega^2\boldsymbol{r}_\perp$ where the second term is centrifugal force.

152. Which side does the Coriolis force deflect objects?

answer. Coriolis force deflects MOVING objects to the RIGHT (with respect to the direction of travel) in the Northern Hemisphere and to the left in the Southern Hemisphere.

153. Coriolis force does not act on the bodies ...

answer. stationary in the rotating frame.

154. Show that the vector $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$ can be written as a potential.

answer. $\mathbf{v} = \nabla\frac{1}{2}\left(ax^2 + by^2 + cz^2\right)$

In particular $A\mathbf{r}$ is conservative.

155. When is a radial vector field conservative?

answer. Always. Let $r = |\mathbf{r}|$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\hat{\mathbf{r}} = \mathbf{r}/|\hat{\mathbf{r}}|$.

A radial vector field is the form $f(r)\hat{\mathbf{r}}$. Such a vector field is conservative if we can find F such that

$$\nabla F(r) = F'(r)\hat{\mathbf{r}} = f(r)\hat{\mathbf{r}}$$

which implies that $F'(r) = f(r)$. Since any continuous function has an anti derivative, we can always find F as long as f is continuous.

Second way to think. The curl of a radial vector field is always zero. Thus if the domain is simply-connected then the vector field must be given by the gradient of a potential.

156. Show that the centrifugal term can be written as a potential.

answer. For $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, let $\boldsymbol{\Omega} = \Omega\mathbf{k}$ be the rotation vector. Let $\mathbf{r}_\perp$ be the perpendicular distance from the axis of rotation $\boldsymbol{\Omega}$, that is $\mathbf{r}_\perp = x\mathbf{i}+y\mathbf{j}$. Since $\boldsymbol{\Omega}\cdot\mathbf{r}_\perp = 0$, $\boldsymbol{\Omega}\times\mathbf{r} = \boldsymbol{\Omega}\times\mathbf{r}_\perp$, and $\boldsymbol{r}_\perp = \frac{1}{2}\nabla\left|\mathbf{r}_\perp\right|^2$

$$-\boldsymbol{F}_{ce} = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}_\perp) = -\Omega^2\boldsymbol{r}_\perp = \nabla\Phi_{ce}$$

Here think $\mathbf{\Omega} = \Omega\mathbf{i}$, $\mathbf{r}_\perp = a\mathbf{j}$ OR use

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Here $\Phi_{ce} = -\frac{1}{2}\Omega^2 |\mathbf{r}_\perp|^2 = -\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}_\perp|^2 = -\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2$.

157. Expand $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

answer. 1st way. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

To recall BAC-CAB rule: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is perpendicular to $\mathbf{B} \times \mathbf{C}$. Thus it lies on the plane spanned by the vectors $\mathbf{B}$ and $\mathbf{C}$.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = b\mathbf{B} + c\mathbf{C}$$

Take the inner product with $\mathbf{A}$ to get

$$b = k\mathbf{C} \cdot \mathbf{A}, \qquad c = -k\mathbf{B} \cdot \mathbf{A}$$

To find k, take $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{j}$, $\mathbf{C} = \mathbf{k}$. Is there an easier way?

2nd expansion. By the Jacobi identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) - \mathbf{B} \times (\mathbf{A} \times \mathbf{C})$$

158. $\frac{\mathrm{D}}{\mathrm{D}t} \int_V \mathrm{d}V = \int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{v} \mathrm{d}V$

2019-07-11

159. No viscosity means * in the fluid are everywhere zero.

answer. *=tangential stresses

160. Learn the nature of viscosity. ToDO

161. **Kelvin's minimum energy theorem**. The steady irrotational incompressible flow in a simply connected region has less kinetic energy than any other motion with the same normal component of velocity at the boundary.

answer. Because of simply-connectedness and irrotational vector fields are gradient, $\mathbf{u} = \nabla\phi$. Let $\mathbf{U}$ be any other field

$$\nabla \cdot \mathbf{U} = 0 \text{ in } D, \qquad \mathbf{U} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \text{ on } \partial D.$$

Then

$$\begin{aligned}
\int_D \rho |\mathbf{U}|^2 &= \int_D \rho(\mathbf{U} - \mathbf{u} + \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u} + \mathbf{u}) \\
&= \int_D \rho |\mathbf{U} - \mathbf{u}|^2 + 2\rho(\mathbf{U} - \mathbf{u}) \cdot \mathbf{u} + \rho\mathbf{u}^2 \\
&\geq \int_D \rho\mathbf{u}^2
\end{aligned}$$

Since gradient fields ($\mathbf{u}$) and divergence free vector fields with zero normal component on the boundary ($\rho(\mathbf{U}-\mathbf{u})$) are orthogonal $\int_D \rho(\mathbf{U}-\mathbf{u})\cdot\mathbf{u} = 0$ proving the claim.

162. Find an irrotational incompressible vector field.

answer. Gradient of a harmonic function such as $\widehat{\mathbf{r}}/r^2$.

163. When is curl a left inverse to the Biot-Savart operator

$$\mathrm{BS}(V)(y) = (1/4\pi)\int_\Omega V(x)\times(y-x)/|y-x|^3 \mathrm{d}\,(\mathrm{vol}_x)$$

answer. The equation $\nabla\times\mathrm{BS}(V) = V$ holds in Ω if and only if V is divergence-free and tangent to the boundary of Ω.

This is an original result https://www.maths.ed.ac.uk/~v1ranick/papers/candetgl.pdf. One side is well known.

164. If $\alpha = dg$, $g: M\to\mathbb{R}$ and $c:[a,b]\to M$ is a path, find $c^*(\alpha)$ and compute $\oint_{c([a,b])}\alpha$.

answer.

$$c^*(\alpha) = c^*(dg) = dc^*(g) = d(g\circ c)$$

Hence

$$\oint_{c([a,b])}\alpha = \oint_{[a,b]} c^*(\alpha) = \oint_{[a,b]} d(g\circ c) = g\circ c(b) - g\circ c(a)$$

165. Find $c^*(\alpha)$ for $c(t) = (\cos t, \sin t)$ and $\alpha = \dfrac{-ydx + xdy}{x^2+y^2}$.

answer. $c^*\,(\alpha) = dt$

166. Properties of pullback on differential forms.

answer.

- linear: $\phi^*(a\alpha + b\beta) = a\phi^*(\alpha) + b\phi^*(\beta)$ for all scalars a and b and all k forms α and β on V ;
- multiplicative: $\phi^*(\alpha\wedge\beta) = \phi^*(\alpha)\wedge\phi^*(\beta)$ for all k -forms α and l -forms β on V;
- respects composition: $\phi^*\,(\psi^*(\alpha)) = (\psi\circ\phi)^*(\alpha)$, where $\psi: V\to W$ is a second smooth map with W open in $\mathbf{R}^l$, and α is a k -form on W .
- Commutes with d: $d\phi^* = \phi^* d$.

167. Let $\phi(x_1, x_2) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}))$. Find $\phi^*(dy_1)$, $\phi^*(dy_1 \wedge dy_2)$. Example take $\phi(x_1, x_2) = (x_1x_2, \sin x_2)$.

<u>answer</u>. $\phi^*(dy_1) = d\phi_1$ and $\phi^*(dy_1 \wedge dy_2) = d\phi_1 \wedge d\phi_2$.

<u>Example</u>. If $\phi(x_1, x_2) = (x_1x_2, \sin x_2)$ then $\phi^*(dy_1) = d(x_1x_2) = x_2dx_1 + x_1dx_2$ and $\phi^*(dy_1 \wedge dy_2) = d(x_1x_2) \wedge d(\sin x_2) = (x_2dx_1 + x_1dx_2) \wedge \cos x_2 dx_2$.

<u>2019-07-10</u>

168. Explain the terms in accleration in a rotating frame.

<u>answer</u>. $\mathbf{a}_\mathrm{r} = \mathbf{a}_\mathrm{i} - 2\mathbf{\Omega} \times \mathbf{v}_\mathrm{r} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \times \mathbf{r}$ where $\mathbf{a}_\mathrm{r} \stackrel{\text{def}}{=} \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_\mathrm{r}$ is the apparent acceleration in the rotating reference frame, the term $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ represents centrifugal acceleration, and the term $-2\mathbf{\Omega} \times \mathbf{v}_\mathrm{r}$ is the Coriolis acceleration. The last term $\left(-\frac{\mathrm{d}\Omega}{\mathrm{d}t} \times \mathbf{r}\right)$ is the Euler acceleration and is zero in uniformly rotating frames.

169. Give a neccessary and sufficient condition for the linear system $\dot{x} = Ax$ to be stable.

<u>answer</u>. The eigenvalues satisfy $\Re\lambda \leq 0$ and any eigenvalue with $\Re\lambda = 0$ has equal algebraic and geometric multiplicities.

170. Solve $\frac{Df(x,t)}{Dt} = 0$.

<u>answer</u>. The solution is $f(x, t) = f_0(\phi^{-1}(x, t))$ for an arbitrary function f_0. To see, let $f(\phi(a, t), t) = f_0(a)$ then $0 = \frac{\partial f_0}{\partial t} = \frac{Df}{Dt}$.

171. What is the no-magnetic monopole law?

<u>answer</u>. Gauss's Law for magnetism: Divergence of magnetic field is zero.

172. What is the relation between Jacobian J of the motion and the deformation gradient $\mathbf{F}$?

<u>answer</u>.

$$J(\mathbf{a}, t) = \det \mathbf{F} = \det \left(\frac{\partial x_i}{\partial a_j}\right) = \det \left(\nabla_\mathbf{a} x_1, \cdots, \nabla_\mathbf{a} x_n\right) \neq 0$$

since the motion is assumed to be invertible.

173. Express dirac delta function as the Laplacian of a scalar function and as the divergence of a vector field in $\mathbb{R}^3$.

<u>answer</u>.

$$\frac{1}{4\pi}\Delta\left(\frac{-1}{r}\right) = \frac{1}{4\pi}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \delta^3(\mathbf{r}).$$

Its integral over whole space is 1 while it is zero everwhere except origin. Here $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$, $\mathbf{r}$ is the position vector and r is its norm.

More generally if $\mathbf{s} = \mathbf{r} - \mathbf{a}$, $\nabla \cdot \left(\frac{\hat{\mathbf{s}}}{|\mathbf{s}|^2} \right) = 4\pi\delta^3(\mathbf{s})$

174. Give the curve definition of tangent vector on a manifold.

<u>`answer`</u>. Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n -manifold M are called equivalent at the point m if

$$c_1(0) = c_2(0) = m$$

$$\frac{d}{dt} (\varphi \circ c_1)|_{t=0} = \frac{d}{dt} (\varphi \circ c_2)|_{t=0}$$

in some chart φ.

A tangent vector v to a manifold M at a point $m \in M$ is an equivalence class of curves at m.

2019-07-09

175. Write $m\mathbf{a} = \mathbf{f}$ in rotating frame.

<u>`answer`</u>.

$$m\mathbf{a}' = \mathbf{f} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - 2m\mathbf{\Omega} \times \mathrm{v}'$$

since

$$\mathbf{a} = \mathbf{a}' + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2\mathbf{\Omega} \times \mathbf{v}'$$

176. Apply the inertial time derivative twice to get the relation between inertial and rotating accelerations.

<u>`answer`</u>. Let d/dt and d/dt' be the time derivatives in inertial and rotating frames respectively. Then

$$\frac{d}{dt} = \frac{d}{dt'} + \mathbf{\Omega}\times$$

Define

$$\mathbf{v}' = \frac{d\mathbf{r}}{dt'}, \qquad \mathbf{a}' = \frac{d\mathbf{v}'}{dt'}$$

Then

$$\mathbf{a} = \left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right)\mathbf{r} = \left(\frac{d}{dt}\right)(\mathbf{v}' + \mathbf{\Omega} \times \mathbf{r})$$

to get

$$\mathbf{a} = \mathbf{a}' + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2\mathbf{\Omega} \times \mathbf{v}'$$

177. Time derivatives in rotating vs inertial frame?

<u>`answer`</u>. Let d/dt and d/dt' be the time derivatives in inertial and rotating frames respectively. Then

$$\frac{d}{dt} = \frac{d}{dt'} + \mathbf{\Omega}\times$$

Note that if the position is stationary in rotating frame $\frac{dr}{dt'} = 0$, then in inertial frame it is rotating with constant angular velocity $\Omega \times r$.

178. Show that if A is a deformation retract of X then A and X are homotopy equivalent.

<u>`answer`</u>. We know the existence of a continuous map

$$H : X \times [0, 1] \to X$$

such that for every x in X and a in A,

$$H(x, 0) = x, \quad H(x, 1) \in A, \quad \text{and} \quad H(a, 1) = a$$

Let $r = H(\cdot, 1) : X \to A$ and $i : A \to X$ be the inclusion map. Then $ri = 1_A : A \to A$ and $ir = r \simeq 1_X$.

179. Define deformation retraction and a deformation retract of a topological space.

<u>`answer`</u>. A deformation retraction is a homotopy between a retraction and the identity map on X . The subspace A is called a deformation retract of X.

Specifically, a continuous map

$$H : X \times [0, 1] \to X$$

is a deformation retraction of a space X onto a subspace A if, for every x in X and a in A,

$$H(x, 0) = x, \quad H(x, 1) \in A, \quad \text{and} \quad H(a, 1) = a$$

180. A deformation retraction is a special case ...

<u>`answer`</u>. of a homotopy equivalence. In fact, two spaces are homotopy equivalent if and only if they are both deformation retracts of a single larger space.

181. Define a retraction (map) and a retract (subspace) of a topological space.

<u>`answer`</u>. Let X be a topological space and A a subspace of X. Then a continuous map $r : X \to A$ is a retraction if $r(a) = a$ for all a in A.

A subspace A is called a retract of X if such a retraction exists.

182. Lebesgue integrable functions are (1) for computational reasons.

<u>`answer`</u>. (1) not important

183. What is the relation between Riemann and Lebesgue integrable functions resemble?

<u>`answer`</u>. To that of real numbers to rational numbers. Concrete calculations require only rational numbers, but it is the completeness of the real number system which makes it powerful.

184. Discuss the difference of sums in Riemann and Lebesgue approach.

<u>`answer`</u>. In the Riemann scheme one partitions the x interval then forms the sum $\sum_1^n f(\xi_k)(x_{k+1} - x_k)$ for arbitrary ξ_k in $[x_k, x_{k+1}]$ and passes to the limit $n \to \infty$.

In the Lebesgue approach it is the y axis that is partitioned. Let E_i be the set of values of x such that $y_i \leq f(x) \leq y_{i+1}$. We then form $\sum_1^n \eta_i m(E_i)$ where $y_i \leq \eta_i \leq y_{i+1}$ is arbitrarily chosen and $m(E_i)$ is the measure of E_i.

185. Find the integral curves and stagnation points of $v(x, y) = (-y, x)$.

<u>`answer`</u>. Counterclockwise circles around origin. The origin is a stationary point.

186. Find the integral curves and stagnation points of $v(x, y) = (y, -x)$.

<u>`answer`</u>. Clockwise circles around origin. The origin is a stationary point.

187. Find the integral curves and stagnation points of $v(x, y) = (x, y)$.

<u>`answer`</u>. Straight half lines emanating outward from origin. $(0, 0)$ is a stationary point.

<u>`2019-07-08`</u>

188. Show that $f : M \to N$ has a pullback $f^* : H^p_{\text{dR}}(N) \to H^p_{\text{dR}}(M)$.

<u>`answer`</u>. Since the pullback takes closed/exact forms on N to closed/exact forms M.

For a closed p-form ω on N, let $[\omega]$ be the equivalance class in $H^p_{\text{dR}}(N)$. Define $f^*[\omega] = [f^*\omega]$. To show that f^* is well-defined, take $\omega_1 \in [\omega]$ that is $\omega_1 = \omega + \alpha$ where α is an exact p-form on N. Since $f^*\omega_1 - f^*\omega = f^*\alpha$ which is an exact p-form on M, it follows that $[f^*\omega] = [f^*\omega_1]$. Thus the definition does not depend on the particular element chosen from the equivalance class.

189. Give the integral representations of grad, curl, div.

answer.

$$\nabla f(x) = \lim_{\text{vol}(V)\to 0} \frac{1}{\text{vol}(V)} \iint_{\partial V} \mathbf{n}(y) f(y) dS_y$$

$$\nabla \cdot \mathbf{F}(x) = \lim_{\text{vol}(V)\to 0} \frac{1}{\text{vol}(V)} \iint_{\partial V} \mathbf{n}(y) \cdot \mathbf{F}(y) dS_y,$$

$$\nabla \times \mathbf{F}(x) = \lim_{\text{vol}(V)\to 0} \frac{1}{\text{vol}(V)} \iint_{\partial V} \mathbf{n}(y) \times \mathbf{F}(y) dS_y,$$

where V is a ball centered at x. Notice that ∇ is replaced by $\mathbf{n}$ inside the integral. Also the definition of divergence is intuitive by the divergence theorem.

2019-07-05

190. $\rho \frac{D\mathbf{u}}{Dt} = \mathbf{s}$ is the Cauchy momentum equation. Write $\mathbf{s}$ as a sum of two terms.

answer.

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$$

where $\boldsymbol{\sigma}$ denotes the molecular forces, and $\rho\mathbf{f}$ denotes the body forces.

191. Cauchy momentum equation has the form $\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$. What is the general form $\boldsymbol{\sigma}$?

answer.

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ is traceless and is called deviatoric part of the stress tensor.

192. Cauchy momentum equation has the form $\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot (-p\mathbf{I} + \boldsymbol{\tau}) + \rho \mathbf{f}$. What is the general form $\boldsymbol{\tau}$ for a Newtonian fluid?

answer.

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \left(\frac{\partial u_l}{\partial x_l} \right) \delta_{ij} + \mathbf{T}$$

where the first two terms are due to Newtonian stress, and $\mathbf{T}$ represents non-Newtonian stress.

193. What is the conservation form of the momentum equation?

answer.

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \mathbf{s}$$

This is equivalent to

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{s}$$

proof. The divergence of the dyad is

$$\nabla \cdot (\mathbf{ab}) = (\nabla \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \cdot \nabla \mathbf{b}$$

Expanding

$$\mathbf{u}\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right) + \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \mathbf{s}$$

By mass continuity (second paranthesis) is zero.

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \mathbf{s}$$

194. Why does the stress tensor need to be symmetric?

answer. Otherwise any small fluid element would suffer infinite angular acceleration. See Roberts - Model emergent dynamics in complex systems, pg. 121 in the pdf.

195. Derive the Cauchy momentum equation.

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T}$$

What do the terms on the RHS describe?

answer.

$$\begin{aligned} \text{Total force (body + surface)} &= \int_V dV \rho \mathbf{f} + \int_{\mathbf{S}} \mathbf{T} \cdot d\mathbf{S} \\ &= \int_V dV (\rho \mathbf{f} + \nabla \cdot \mathbf{T}) \end{aligned}$$

where the first integral denotes the long ranged body forces while the second one denotes the short ranged molecular forces, internal to the fluid.

Cauchy momentum equation is just Newton's second law.

196. What is the rate of change of momentum of a material fluid element V_t?

answer.

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{v} dV = \int_{V_t} \left(\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v})\right) dV = \int_{V_t} \rho \frac{D\mathbf{v}}{Dt} dV$$

Since

$$(\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v})) = \mathbf{v} \partial_t \rho + \rho \partial_t \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) + \rho \mathbf{v} \nabla \cdot \mathbf{v}$$

first and third terms cancel due to continuity equation.

197. What are the characteristics of two types of forces?

answer. $\int_V \rho \mathbf{f} dV$: Long ranged external body forces that penetrate matter.

$\int_{\partial V} \mathbf{T} \cdot \hat{\mathbf{n}} dS$: Short ranged molecular forces, internal to the fluid. For any element, the net effect of these due to interactions with other elements acts in a thin surface layer.

2019-07-04

198. What is the stress tensor in a static fluid?

answer. $\sigma = -pI$ where $p = \frac{1}{3}\operatorname{tr}(\sigma)$.

p is called the static pressure.

199. For a static fluid, why is the only stress is the normal stress?

answer. Since by definition a fluid subjected to a shear stress must deform and undergo motion.

200. What is the stress tensor in a moving fluid?

answer. $\sigma = -p\mathbf{I} + \boldsymbol{\tau}$ where $\boldsymbol{\tau}$ is the deviatoric part.

201. What are examples of body forces?

answer. gravity, electric and magnetic forces, fictitious forces such as the centrifugal force, Euler force, and the Coriolis force.

202. The relation between the total force density $\mathbf{f}$, volume force density $\mathbf{F}$, stress tensor σ?

answer. $\int_V f_i dV = \int_V F_i dV + \oint_S \sigma_{ij} dS_j$, in differential form $\mathbf{f} = \mathbf{F} + \nabla \cdot \sigma$.

203. $\nabla \times (\mathbf{A} \times \mathbf{B})$?

answer. $= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

204. The incompressible velocity u satisfies the no-flow condition $u \cdot n = 0$ on ∂D (with n the unit

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)dy$$

where the Kernel K_D is given by $K_D(x,y) := \nabla^\perp G_D(x,y)$, with $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and $G_D \geq 0$ the Green's function for D.

2019-07-03

answer.

205. Define the derivative of a map between two manifolds, $f : M \to N$ at a point M using the curve definition.

answer. It is the linear map $df : TM \to TN$ constructed such that $df(\frac{d\gamma}{dt}(0)) = \frac{d}{dt}(f \circ \gamma)(0)$ where γ is a curve $\gamma(0) = m$ and $\frac{d\gamma}{dt}(0)$ is its tangent vector at m.

206. What is the temporal variability of thermocline?

answer. The thermocline structure in the major ocean basins can be broadly thought of as a "permanent" or "long-term mean" temperature profile with a superimposed seasonal variation that mainly affects the upper mixed layer, including its thickness.

207. What is the spatial variability of thermocline in oceans?

answer. The actual profile varies greatly from location to location. In shallower waters, the seasonal variation is more pronounced and may dominate the profile. At the higher latitudes, the temperature difference is much smaller and the thermocline disappears altogether about 60 degrees latitude.

208. Triple product with Levi-civita?

answer.

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \epsilon_{ijk} u_i v_j w_k$$

209. Define Levi-Civita symbol as a triple product.

answer.

$$\epsilon_{ijk} = \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)$$

210. What is the evolution equation of the probability density function $\Theta(x, t)$ which is associated with the Langevin equation?

answer. The Fokker-Planck equation

$$\partial_t \Theta + \boldsymbol{\partial} \cdot (\boldsymbol{v} \Theta) = \kappa \partial^2 \Theta$$

211. What is the equation of particles advected by a velocity field v and subject to molecular diffusion?

answer. The Langevin equation

$$\frac{d\boldsymbol{x}(\boldsymbol{a}, t)}{dt} = \boldsymbol{v}(\boldsymbol{x}, t) + \boldsymbol{W}(t)$$

The function $\boldsymbol{x}(\boldsymbol{a}, t)$ denotes the position at time t of the particle which was in in $\boldsymbol{a}$. The random process $\boldsymbol{W}(t)$ is Gaussian, independent of $\boldsymbol{v}$, has zero mean a white-noise in time. The constant κ appearing is the molecular diffusivity.

$$E\left(W_i(t)W_j\left(t'\right)\right) = 2\kappa\delta_{ij}\delta\left(t - t'\right)$$

212. How to remember the notation for pullback and pushforward?

<u>answer</u>. You can push-forward the the tangent space for any morphism of smooth manifolds, and you can pull-back the cotangent space. The cotangent bundle is usually written using the star for dual as T^*M. Ergo, pull-backs by smooth maps are f^*. Also usually upper starred objects map in reverse direction such as adjoint operators.

<u>2019-07-01</u>

213. Find the unique decomposition of $n \times n$ matrix as a sum of skew-symmetric matrix, a symmetric traceless matrix and a symmetric traceful matrix proportional to the identity.

<u>answer</u>. Given A, write $A = S + R$ with $S = \frac{1}{2}(A + A^T)$ and $R = \frac{1}{2}\left(A - A^T\right)$. Then write $S = Q + T$ where $Q = (S - \frac{1}{n}\operatorname{tr}(S)I)$ is traceless and $T = \frac{1}{n}\operatorname{tr}(S)I$. Since $\operatorname{tr}(S) = \operatorname{tr}(A)$,

$$A = R + T + Q$$

214. Which direction does the earth rotate?

<u>answer</u>. Eastward

215. What does Coriolis parameter represent?

<u>answer</u>. The vertical component of the planetary vorticity.

216. Why is Coriolis parameter $f = 2\mathbf{\Omega}\sin\phi$?

<u>answer</u>. Coriolis parameter is the vertical component of the planetary vorticity $2\mathbf{\Omega}$. $f = 2\mathbf{\Omega}\cdot\widehat{\mathbf{r}} = 2\Omega\sin\phi$ where ϕ is the latitude ($\pi/2$ at the north pole and 0 at the equator).

217. What is the relation between the vertical components of absolute vorticity, rotating vorticity and the Coriolis parameter?

<u>answer</u>. $\eta = \zeta + f$ where η is the vertical component of the absolute vorticity, ζ is the vertical component of the rotating vorticity and f is the Coriolis parameter.

<u>To see</u>, By (AbsVor-RelVor),

$$\boldsymbol{\omega}_I = \boldsymbol{\omega}_R + 2\mathbf{\Omega}$$

The vertical components are $\eta = \boldsymbol{\omega}_I\cdot\widehat{\mathbf{r}}$, $\zeta = \boldsymbol{\omega}_R\cdot\widehat{\mathbf{r}}$ and $f = 2\mathbf{\Omega}\cdot\widehat{\mathbf{r}} = 2\Omega\sin\phi$ where ϕ is the latitude ($\pi/2$ at the north pole and 0 at the equator)

218. Describe projective spaces.

answer. Every time we have a problem involving only the directions of vectors and in which their lengths are irrelevant, we are involved with a projective space. Given an n-dimensional vector space V over the field K, its corresponding projective space KPn is the space formed by all the 1-dimensional subspaces of V . Each point of KPn is the set formed by a vector v and all the vectors proportional to v. We may be in a finite-dimensional vector space, or in an infinite-dimensional space like a Hilbert space.

219. Show that the centripetal force $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a potential force.

answer. Let $\Phi = \frac{1}{2} |\boldsymbol{\omega} \times \mathbf{r}|^2$. Using

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

with $\mathbf{A} = \mathbf{B} = \boldsymbol{\omega} \times \mathbf{r}$ becomes

$$\nabla\Phi = \boldsymbol{\omega} \times \mathbf{r} \times (\nabla \times (\boldsymbol{\omega} \times \mathbf{r})) + (\boldsymbol{\omega} \times \mathbf{r} \cdot \nabla)(\boldsymbol{\omega} \times \mathbf{r})$$

???

220. The time derivative of a scalar quantity is invariant with respect to the inertial and rotating coordinate systems, $\frac{dT}{dt} = \left(\frac{dT}{dt}\right)_r$.

answer. See Dolzhansky page 50 for the proof.

221. Taking the time derivative and assuming that the angular acceleration of earth to be zero, i.e. $\frac{d\vec{\Omega}}{dt} = 0$,

$$\begin{aligned}
\left(\frac{d\vec{q}_I}{dt}\right)_I &= \left(\frac{d\vec{q}_R}{dt}\right)_I + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_I \\
&= \left(\frac{d\vec{q}_R}{dt}\right)_R + \vec{\Omega} \times \vec{q}_R + \vec{\Omega} \times \left[\left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r}\right] \\
&= \left(\frac{d\vec{q}_R}{dt}\right)_R + \underbrace{2\vec{\Omega} \times \vec{q}_R}_{\text{Coriolis acc.}} + \underbrace{\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{centripetal}}
\end{aligned}$$

The centripetal force may be written in terms of a centripetal force potential ϕ_c where

$$\phi_c = \frac{1}{2}(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) = \frac{1}{2}|\Omega|^2 \mathbf{r}_\perp^2$$

$$-\nabla\phi_c = -\frac{d\phi_c}{d\mathbf{r}_\perp}\vec{e}_\perp = -|\Omega|^2 \mathbf{r}_\perp = \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

In the coordinate system rotating at the constant angular velocity, the momentum equation reads, after dropping subscripts R

$$\rho\left(\frac{d\vec{q}}{dt} + 2\vec{\Omega}\times\vec{q}\right) = -\nabla p + \rho\nabla\left(\phi_g + \phi_c\right) + \mu\nabla^2\vec{q}$$

where

$$\phi_g = gz \qquad \phi_c = \frac{1}{2}(\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\vec{r})$$

222. Show that

$$\frac{d}{dt} = \left(\frac{d}{dt}\right)_R + \mathbf{\Omega}\times$$

answer. Let $\mathbf{A} = A_i\mathbf{e}_i$ and use $\frac{d\mathbf{e}_i}{dt} = \mathbf{\Omega}\times\mathbf{e}_i$ to get

$$\frac{d\mathbf{A}}{dt} = \frac{dA_i}{dt}\mathbf{e}_i + A_i\frac{d\mathbf{e}_i}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_R + \mathbf{\Omega}\times\mathbf{A}.$$

223. Derive the relation between velocity vector in inertial and rotating frames.

answer.

$$\mathbf{u}_I = \mathbf{u}_R + \Omega\times\mathbf{r}$$

$$\mathbf{r} = r_i\mathbf{e}_i \implies \frac{d\mathbf{r}}{dt} = \mathbf{u}_I = \frac{dr_i}{dt}\mathbf{e}_i + r_i\frac{d\mathbf{e}_i}{dt} = \mathbf{u}_R + r_i\Omega\times\mathbf{e}_i = \mathbf{u}_R + \Omega\times\mathbf{r}$$

224. State the relation between acceleration in rotating (with constant-angular velocity) and inertial frames.

answer.

$$\frac{d\mathbf{u}}{dt} = \left(\frac{d\mathbf{u}_r}{dt}\right)_r + 2\mathbf{\Omega}\times\mathbf{u}_r + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$$

Absolute acceleration = Relative acc. + Coriolis acc. + centripetal acc.

225. Show that the relation between acceleration in rotating (with constant-angular velocity) and inertial frames is given by

$$\frac{d\mathbf{u}}{dt} = \left(\frac{d\mathbf{u}_r}{dt}\right)_r + 2\mathbf{\Omega}\times\mathbf{u}_r + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$$

answer. Differentiate $\mathbf{u} = \mathbf{u}_r + \mathbf{\Omega}\times\mathbf{r}$ to get

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}_r}{dt} + \mathbf{\Omega}\times\mathbf{u}$$

Use $\frac{d\mathbf{u}_r}{dt} = \left(\frac{d\mathbf{u}_r}{dt}\right)_r + \mathbf{\Omega}\times\mathbf{u}_r$ and $\mathbf{\Omega}\times\mathbf{u} = \mathbf{\Omega}\times(\mathbf{u}_r + \mathbf{\Omega}\times\mathbf{r})$.

226. Why can't we always conclude $\operatorname{curl}\mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$ from Green's Theorem?

answer. The region may not be simply connected. Otherwise the result is true.

Suppose that $\operatorname{curl}\mathbf{F} = 0$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_R \operatorname{curl}\mathbf{F} \cdot \mathbf{n} dS = 0$$

But if R is an annular region then its boundary C consists of multiple curves. Hence this does not imply that $\mathbf{F}$ is independent of path.

2019-06-29

227. What is the fundamental group of the circle? Why?

answer. The additive group of integers. Each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around m+n times. So the fundamental group of the circle is isomorphic to the additive group of integers.

228. Is the fundamental group a topological invariant?

answer. Yes. Homeomorphic topological spaces have the same fundamental group.

229. The first homotopy group is

answer. Fundamental group

230. Define tangent vector on a manifold using curves.

answer. A tangent vector is an equivalence of curves. Two curves $c_i : I \subset \mathbb{R} \to M$, $i = 1, 2$ are equivalent if $c_1(0) = c_2(0) = p \in M$ and for **some** coordinate $x : M \to \mathbb{R}^n$, $(x \circ c_1)'(0) = (x \circ c_2)'(0)$.

It turns out that the definition is independent of the coordinate chosen since if the velocities of the curves are equal for one coordinate, it is equal in every coordinates.

231. Show that the curve definition of tangent vector is independent of the coordinates chosen.

answer. We need to show that if velocity vector in $\mathbb{R}^n$ of two curves are equal for some coordinate then they are equal for any coordinates. Proof follows from chain rule.

A tangent vector is an equivalence of curves. Two curves $c_i : I \subset \mathbb{R} \to M$, $i = 1, 2$ are equivalent if $c_1(0) = c_2(0) = p \in M$ and for **some** coordinate $x : M \to \mathbb{R}^n$, $(x \circ c_1)'(0) = (x \circ c_2)'(0)$.

For if y is another chart near p, then

$$(y \circ c_i)'(0) = (y \circ x^{-1} \circ x \circ c_i)'(0) = (D_{\mathbf{a}} f)\mathbf{v}$$

where $\mathbf{a} = x \circ c_i(0)$, $f = y \circ x^{-1}$, and $\mathbf{v} = (x \circ c_i)'(0)$.

232. Find a non-constant density and incompressible velocity profile satisfying the continuity equation showing that incompressible flow does not mean constant density.

answer. Suppose $\mathbf{u} = \mathbf{i}$, constant so that it is divergence free and $\rho(t = 0) = \rho_0(\mathbf{x})$. Then the continuity equation gives $\rho_t + \rho_x = 0$ which has the non-constant solution $\rho(x, y, z, t) = \rho_0(x - t, y, z)$.

2019-06-28

233. What can be said about the eigenvalues of an orthogonal matrix?

answer. They lie on the unit circle.

234. Prove that the eigenvalues of an orhogonal matrix lie on the unit circle.

answer.

$$Ax = \lambda x \implies |\lambda|^2 |x|^2 = |Ax|^2 = (x^T A^T)(Ax) = x^T x = |x|^2$$

235. How does the deviatoric and hydrostatic parts of the stress tensor show up in NSE?

answer. We write the stress tensor as

$$\sigma = -pI + T$$

where $-pI$ is the hydrostatic stress (which tends to change the volume of the fluid parcel) and T is the deviatoric part (which tends to distort the fluid parcel). Then NSE becomes

$$\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \sigma = -\nabla p + \nabla \cdot T + \vec{f}$$

236. Conservation law for phase space probability?

answer. $f(x_0)dV(x_0) = f(x_t, t)dV(x_t)$.

237. What is Liouville's equation for a Hamiltonian system?

answer. If f is constant on the trajectories (a first integral) then it satisfies

$$f_t + \{f, H\} = 0$$

Alternatively

$$f_t + \dot{x} \cdot \nabla f = 0$$

Classically, $f = \rho$ is the density of the particles.

238. What is the explicit relation (not differential one) between divergence of velocity $\mathbf{u}$ and the Jacobian j of the motion?

answer.

$$j(\phi(a,t),t) = \exp\left(\int_0^t \operatorname{div} \mathbf{u}(\phi(a,s),s)ds\right)$$

To see,

$$\frac{\partial J(a,t)}{\partial t} = J(a,t)(\operatorname{div} \mathbf{U}(a,t))$$

$$J(a,t) = J(a,0)\exp\left(\int_0^t \operatorname{div} \mathbf{U}(a,s)ds\right)$$

Since $J(a,0) = 1$

$$j(\phi(a,t),t) = \exp\left(\int_0^t \operatorname{div} \mathbf{u}(\phi(a,s),s)ds\right)$$

2019-06-27

239. Is a homology sphere simply connected?

answer. Not neccessarily. Having the same homolgy groups does not imply simply-connectedness, only that its fundamental group is perfect (see Hurewicz theorem)

240. What is a homology sphere?

answer. It is a closed n-manifold with the same homology groups of every order as the n-sphere.

241. What is the history of Poincaré conjecture?

answer. In a 1900 paper, Poincaré conjectured that the Betti numbers and torsion coefficients (also known today as homology) could tell you whether or not a space was a sphere.

A few years later, Poincaré showed that he was wrong. He came up with the first of what are called homology spheres: spaces that have the same homology as spheres but are not topologically equivalent to spheres.

Poincaré's discovery of a homology sphere led him to refine his conjecture to what is now known as the Poincaré conjecture. He added another invariant, known as the fundamental group, and believed that if a manifold had the same homology and fundamental group as a sphere, it had to be a sphere. Poincaré used the fundamental group of the homology sphere to show that it was topologically different from a sphere.

The Poincaré conjecture was one of the most important unsolved conjectures In 2006, this conjecture was finally proved, with Russian mathematician Grigori Perelman putting the finishing touches on the proof.

242. What is Poincaré conjecture?

<u>answer</u>. Every simply-connected, closed (compact without boundary) 3-manifold is homeomorphic to a 3-sphere proved a century later by Perelman.

2019-06-26

243. In which domains, a divergence-free vector field a curl? Give special examples.

<u>answer</u>. When the second de-Rham cohomology vanishes. In particular, if the domain is contractible (all de-Rham cohomologies vanish) then any divergence free vector field is a curl.

244. Show that in a star shaped domain, the curl can be constructed explicitly.

<u>answer</u>. ToDO.

See https://www.math.unl.edu/~mbrittenham2/classwk/208s04/inclass/divergence-frees_are_curls.pdf

245. 3 term decomposition of velocity gradient?

<u>answer</u>.

$$\mathbf{L} = \left(\frac{1}{3}(\operatorname{div}\mathbf{u})\mathbf{I}\right) + \left(\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right) - \frac{1}{3}(\operatorname{div}\mathbf{u})\mathbf{I}\right) + \left(\frac{1}{2}\left(\mathbf{L}-\mathbf{L}^T\right)\right)$$

(traceful) isotropic component + (traceless) rate of deformation tensor + vorticity tensor

246. Velocity gradient as a dyadic?

<u>answer</u>. $\nabla\mathbf{u} \equiv \nabla \otimes \mathbf{u}$

247. What is Schur decomposition?

<u>answer</u>. If A is a complex square matrix, then $A = UTU^*$ where T is triangular and U is unitary.

248. (RBCAV) What is the relation between the circulation Γ and the vorticity ω?

answer.

$$\Gamma = \int_C \mathbf{u} \cdot d\mathbf{x} = \int_D \boldsymbol{\omega} \cdot \mathbf{n} dS$$

for any surface D having the closed curve C as its boundary.

249. The circulation of the velocity over a curve is the (*) the vorticity (**).

answer. (*) flux of (**) through a surface containing that curve. See (RBCAV).

250. What is absolute vorticity?

answer. It is the curl of absolute velocity (inertial reference frame).

$$\boldsymbol{\omega}_I = \nabla \times \mathbf{v}_I$$

251. (AbsVor-RelVor) Find the relation between the absolute vorticity and relative vorticity.

answer.

$$\boldsymbol{\omega}_I = \nabla \times \mathbf{v}_I = \boldsymbol{\omega}_R + 2\boldsymbol{\Omega}$$

To see,

$$\mathbf{v}_I = \mathbf{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}$$

The result follows from the simplification

$$\nabla \times (\boldsymbol{\Omega} \times \mathbf{r}) = \boldsymbol{\Omega}(\nabla \cdot \mathbf{r}) - \boldsymbol{\Omega} \cdot \nabla \mathbf{r} = 3\boldsymbol{\Omega} - \boldsymbol{\Omega} = 2\boldsymbol{\Omega}.$$

252. If M is simply-connected then first de-Rham cohomology of M vanishes. Is the converse true?

answer. Only in some cases such as when M is a domain in $\mathbb{R}^2$. But in general the converse does not imply that the first fundamental group is trivial. It only means that there are no nontrivial homomorphisms from the fundamental group to real numbers.

https://math.stackexchange.com/questions/1689092.

253. Show that when M is simply-connected, any closed 1-form is exact.

answer. In a simply-connected space, any closed path is homotopic to a point and since $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ for closed 1-form and homotopic paths, it follows that ω is exact.

254. In a simply-connected manifold, closed 1-forms are exact. Is this a necessary condition?

answer. No. Homology spheres have vanishing first de-Rham cohomology which have non-trivial fundamental group.

255. What is the condition on 1-form ω and curves γ_1, γ_2 so that $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$?

answer. γ_0 and γ_1 are homotopic and ω is a closed form.

256. Prove that if γ_0 and γ_1 are homotopic and $\omega = P(x, y)dx + Q(x, y)dy$ is a closed form then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

This result obviously generalizes to n-dimensional case.

answer. Let $\gamma_t(s) = (x(t, s), y(t, s))$, $t \in [0, 1]$, $s \in [0, 1]$ be a homotopy between γ_0, γ_1 so that $\gamma_t(0) = a$, $\gamma_t(1) = b$ for all t. Then

$$\begin{aligned}
\frac{d}{dt} I(t) &= \frac{d}{dt} \oint_{\gamma_t} P(x, y)dx + Q(x, y)dy \\
&= \int_0^1 \frac{\partial}{\partial t} \left(P(\gamma_t(s)) \frac{\partial x}{\partial s} + Q(\gamma_t(s)) \frac{\partial y}{\partial s} \right) ds \\
&= \int_0^1 \frac{\partial}{\partial s} \left(P(\gamma_t(s)) \frac{\partial x}{\partial t} + Q(\gamma_t(s)) \frac{\partial y}{\partial t} \right) ds
\end{aligned}$$

The above change of order of differentiation uses that $P_y = Q_x$. Hence

$$\frac{d}{dt} I(t) = P(b) \frac{\partial x}{\partial t}(t, 1) + Q(b) \frac{\partial y}{\partial t}(t, 1) - P(a) \frac{\partial x}{\partial t}(t, 0) + Q(a) \frac{\partial y}{\partial t}(t, 0)$$

Since $x(t, 1)$ is constant in t, $\frac{\partial x}{\partial t}(t, 1) = 0$. Similarly the other partial derivatives are also zero.

2019-06-25

257. Formula for the Coriolis frequency f?

answer. The Coriolis frequency $f = 2\Omega \sin \varphi$ also called the Coriolis parameter or Coriolis coefficient, is equal to twice the rotation rate Ω of the Earth multiplied by the sine of the latitude ϕ.

2019-06-24

258. What are the two essential elements of integration by parts of differential forms on manifolds?

answer. Integration by parts for differential forms boils down to Leibniz rule $\mathrm{d}(\eta \wedge \omega) = \mathrm{d}\eta \wedge \omega + (-1)^{\mathrm{degree}(\eta)} \eta \wedge \mathrm{d}\omega$ and the Stokes Theorem $\int_M \mathrm{d}\omega = \int_{\partial M} \omega$.

259. If you shot a missile due north from near the equator, how would an observer from (A) outer space, (B) on earth observe the motion?

<u>`answer`</u>. (A) To an observer in outer space, the missile would appear to have traveled in a straight line in NE direction (velocity = velocity(Earth) + vel(Missile)), (B) to an observer on earth the missile would appear to have curved to the right.

Equator moves faster. The missile would carry with it the eastward momentum from that latitude. As it traveled northward above the earth it would retain that eastward momentum while the earth below it would have less eastward momentum, so it would land at a longitude east of where it was launched.

260. Is there a stable equilibrium in any electrostatic field?

<u>`answer`</u>. The following argument is not quite right. Read Earnshaw's Theorem. ToDO

Only if a charge is right on top of another charge.

P0 is a stable location, if for any sphere around P0, the force field is inward. By Gauss Law, it means there must be a charge at P0.

261. If $w = \int^x f(\lambda)d\lambda$ then find dw.

<u>`answer`</u>. $f(x)dx$.

262. What are the basic consequences of

$$\frac{\partial J(a,t)}{\partial t} = J(a,t)\,\mathrm{div}\left(\frac{\partial \phi(a,t)}{\partial t}\right) = J(a,t)U(a,t)$$

<u>`answer`</u>.

It implies Reynolds transport theorem

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)dV_x = \int_{\Omega_t}\left(\frac{D}{Dt} + \mathrm{div}\left(\frac{\partial \phi(a,t)}{\partial t}\right)\right) f(x,t)dV_x$$

In particular, if the divergence of the velocity of the motion is zero (e.g. Hamiltonian systems), then the volumes in phase space are conserved.

263. State the evolution equation for the (Eulerian) Jacobian of the motion $x = \phi(a,t)$.

<u>`answer`</u>.

$$\frac{D}{Dt} j(x,t) = j(x,t)\,\mathrm{div}\, u(x,t)$$

where

$$J(a,t) = \det\left(\nabla_a \phi(a,t)\right), \qquad j(x,t) = J(\phi^{-1}(x,t),t),$$

$$u(x,t) = U(\phi^{-1}(x,t),t), \qquad U(a,t) = \frac{\partial \phi(a,t)}{\partial t}$$

264. State the evolution equation for the (Lagrangian) Jacobian of the motion $x = \phi(a,t)$.

<u>answer</u>.

$$\frac{\partial J(a,t)}{\partial t} = J(a,t)\,\mathrm{div}\,U(a,t),$$

where

$$J(a,t) = \det\left(\nabla_a \phi(a,t)\right), \qquad U(a,t) = \left(\frac{\partial \phi(a,t)}{\partial t}\right)$$

265. What is the index of origin for the vector field z^n, for n an integer? Why?

<u>answer</u>. n because on any circle around origin, the vector field $z \to z^n = r^n \exp(in\theta)$ rotates n times counterclockwise if n is positive and clockwise if n is negative.

266. Can non-degenerate critical points of a function be non-isolated? Why?

<u>answer</u>. No. By Morse Lemma, near a non-degenerate critical point, the function behaves like a quadratic polynomial whose gradient field is non-vanishing except at the singular point.

267. Outline the proof that de Rham cohomology is invariant under homotopy equivalance.

<u>answer</u>. (1) Let $f : M \to N$. Then f^* (pullback map) maps closed (resp. exact) forms on N to closed (resp. exact) forms on M. Thus induces a linear map, still denoted f^*, $f^* : H^k_{de}(N) \to H^k_{de}(M)$.

(2) Prove a generalization of Poincare's lemma that states that if $f, g : M \to N$ are homotopic then $f(\omega) - g(\omega)$ is an exact form. Therefore, they induce the same map from $H^k_{de}(N) \to H^k_{de}(M)$. (The proof of this is almost exactly the same as the proof of Poincare's lemma.)

(3) Let M and N be homotopic manifolds. Let $f : M \to N$ and $g : N \to M$ be functions so that $f \circ g$ and $g \circ f$ are homotopic to the identity map. Then $(f \circ g)^* = g^* \circ f^*$ induces the identity map on H^k_{de} and similarly for $(g \circ f)^*$. Therefore f^* and g^* are inverses of each other and give a bijection between $H^k_{de}(M)$ and $H^k_{de}(N)$.

268. De Rham Cohomology is additive, which means for $\{M_i\}_{i \in I}$ smooth manifolds that;

$$H^n_{\mathrm{dR}}\left(\coprod_{i \in I} M_i\right) \cong \bigoplus_{i \in I} H^n_{\mathrm{dR}}\left(M_i\right).$$

<u>answer</u>. https://www.math.vu.nl/~vdvorst/DeRham.pdf.

269. Define the contraction of a differential k-form.

answer. For a vector field X, $i_X : \Gamma^k \to \Gamma^{k-1}$,

$$i_X \theta := \theta(X, \cdot, \cdots, \cdot)$$

270. What are the 3 basic properties of contraction (interior product) of a differential form?

answer. (1) i_X is linear in X, (2) $i_X \circ i_X = 0$, (3) $i_X(\sigma \wedge \omega) = (i_X \sigma) \wedge \omega + (-1)^{\deg(\sigma)} \sigma \wedge (i_X \omega)$.

271. Compute directly the flux of $\hat{\mathbf{r}}/r^2$ over a sphere of radius R. Will divergence theorem work?

answer.
$$\begin{aligned}\oint \mathbf{v} \cdot d\mathbf{a} &= \int \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot \left(R^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}\right) \\ &= \text{area of unit sphere} = 4\pi\end{aligned}$$

The divergence theorem will not work out of the box. $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$. To use divergence theorem, we need to consider a small sphere around the origin. Then divergence theorem says that the flux out of the big sphere equals the negative of the flux out of the small sphere. The flux out of the small sphere can be computed using the mean value theorem for integrals.

Alternatively, $\frac{\mathbf{r}}{r^3} = -\nabla(1/r)$ so applying divergence theorem,

$$\oint \mathbf{v} \cdot d\mathbf{a} = \int_{B_R} -\Delta\left(\frac{1}{r}\right) dV = \int_{B_R} 4\pi\delta(\mathbf{r}) dV = 4\pi$$

2019-06-23

272. If two spaces are homotopy-equivalent, are they homeomorphic as well?

answer. Any contractible space is by definition homotopy equivalent to the one-point space, for instance an interval, a disk, the real line, the Euclidean plane. Any space with more than a single point is not homeomorphic to the one-point space because cardinality is a homeomorphism invariant.

273. What is the relation between the Euler characteristic of a compact manifold and the Morse index of an arbitrary real valued function defined on that manifold?

answer. Take any smooth function $f : N \to \mathbb{R}$ on the compact manifold N. If n_k denotes the number of critical points with index k, then

$$\sum_{i=0}^{n} (-1)^k n_k = \chi(N) = \text{the Euler characteristic of } N.$$

Note that the sum is independent of the function chosen.

274. What is the relation between index and topological degree?

<u>answer</u>. For $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, if x_0 is an isolated zero then

$$\operatorname{ind}(v, x_0) := \deg(v, B, 0)$$

where B is a sufficiently small neighborhood of x_0 which contains no other zeros of v.

By the properties of topological degree, this definition does not depend on the choice of B.

Index is local topological degree which does not see other zeros. Or topological degree is global index.

275. For $v : \mathbb{R}^n \to \mathbb{R}^n$, if $v(x_0) = 0$ and $\det(Dv(x_0)) \neq 0$ then find $\operatorname{ind}(v, x_0)$ in terms of eigenvalues of $Dv(x_0)$.

<u>answer</u>. $\operatorname{ind}(v, x_0) = (-1)^m$ where m is the number of negative eigenvalues of $Dv(x_0)$. This follows from the property

$$\deg(v, B, 0) = \sum_{x \in f^{-1}(0)} \operatorname{sgn} \det Dv(x)$$

For $x_0 \subset B$ with B sufficiently small, RHS is $\operatorname{sgn} \det Dv(x_0)$ and LHS is $\operatorname{ind}(v, x_0)$.

276. Show that $\chi(S^2) = 2$.

<u>answer</u>. Take the z-projection function $z = \pm \left(1 - x^2 - y^2\right)^{1/2}$ on S^2 where sign is positive/negative in the upper/lower semi-sphere. Then the critical points of this function are the north pole which is a max and thus have index 2, and the south pole which is a min and thus have index 0.

277. On which manifolds, any vector field must have a singular point?

<u>answer</u>. On compact manifolds with non-zero Euler-Poincare characteristic.

278. If M is a 3-manifold which has the same homology groups as the 3-sphere, then is M homeomorphic to 3-sphere?

<u>answer</u>. No! There is a manifold, called 3-homology sphere, which has a non-trivial fundamental group but has the same homology groups as the 3-sphere. Since fundamental group is a topological invariant, 3-homology sphere is not homeomorphic to 3-sphere.

Poincare himself constructed this manifold and showed that homology is not enought to guarantee homeomorphism.

279. State the Poincaré conjecture.

answer. Every simply-connected, closed (compact without boundary) 3-manifold is homeomorphic to a 3-sphere proved a century later by Perelman.

2019-06-22

280. Integrate by parts $\int_0^x (t-x)f'(t)dt$.

answer. $\int_0^x (t-x)f'(t)dt = (t-x)f(t)\mid_{t=0}^x - \int_0^x f(t)dt = xf(0) - \int_0^x f(t)dt$.

281. What is the standard flat Riemannian metric?

answer. $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ on $\mathbb{R}^3$.

282. In general, a re-parametrization of an integral curve is no longer an integral curve. If $\gamma : I \to M$ is an integral curve of X, find a canonical re-parametrization of γ which is still an integral curve of X.

answer. Let $I_a = \{t | t+a \in I\}$ and $\gamma_a(t) := \gamma(t+a)$, then $\gamma_a : I_a \to M$ is an integral curve of X

283. If $\gamma : I \to M$ is an integral curve of X find an integral curve of aX where a is a constant?

answer. Let $I^a = \{t | at \in I\}$ and $\gamma^a(t) := \gamma(at)$, then $\gamma^a : I^a \to M$ is an integral curve for $X^a = aX$.

284. Consider the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Find the integral curves.

answer. If $\gamma(t) = (x(t), y(t))$ is an integral curve of X, we must have for any $f \in C^\infty\left(\mathbb{R}^2\right)$

$$x'(t)\frac{\partial f}{\partial x} + y'(t)\frac{\partial f}{\partial y} = \nabla f \cdot \frac{d\gamma}{dt} = X_{\gamma(t)}f = x(t)\frac{\partial f}{\partial y} - y(t)\frac{\partial f}{\partial x}$$

which is equivalent to the system

$$x'(t) = -y(t), \quad y'(t) = x(t)$$

The solution to this system is

$$x(t) = a\cos t - b\sin t, \quad y(t) = a\sin t + b\cos t$$

These are circles centered at the origin in the plane parametrized by the angle (with counterclockwise orientation).

285. Write the differential equations associated with $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}^n$. Find the integral curves of X.

<u>`answer`</u>. The straight lines parallel to the x^1- axis, parametrized as $\gamma(t) = (c_1 + t, c_2, \cdots, c_n)$. To check this, we need to show that

$$\dot{\gamma}(f) = X_\gamma(f),$$

for any smooth function f on $\mathbb{R}^n$.

$$\dot{\gamma}(f) = \frac{d}{dt}(f \circ \gamma) = \nabla f \cdot \frac{d\gamma}{dt} = \frac{\partial f}{\partial x^1} = X_\gamma(f)$$

In coordinates the integral curve satisfies

$$\dot{x_1} = 1, \qquad \dot{x_2} = 0, \cdots \quad \dot{x_n} = 0$$

286. We say that a smooth curve $\gamma : I \to M$ is an integral curve of X if

<u>`answer`</u>. For any $t \in I$, $\dot{\gamma}(t) = X_{\gamma(t)}$. That is $\frac{d}{dt}(f \circ \gamma)(t) = X_{\gamma(t)}(f)$ for every $f : M \to \mathbb{R}$.

287. Let $\gamma : I \to M$ be a curve. Define the tangent vector of γ at the point $\gamma(0)$ as a derivation.

<u>`answer`</u>. For $f : M \to \mathbb{R}$, $\dot{\gamma}(0)(f) = \frac{d}{dt}(f \circ \gamma)(0)$.

288. What is the pushforward of a map between subsets of Euclidean spaces?

<u>`answer`</u>. Between Euclidean spaces, the pushforward of a map is simply the Jacobian.

289. What is the pushforward of a map between two manifolds?

<u>`answer`</u>. Its differential which maps tangent vectors at a point x to a tangent vector at f(x).

290. What are the two laws of electrostatics?

<u>`answer`</u>. (1) Gauss' Law: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, (2) Faraday's Law: $\nabla \times \mathbf{E} = \mathbf{0}$.

(2) follows from Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Remark: The second law says that the steady electric field in a simply connected domain has a potential.

291. The relation between pressure and stress?

<u>`answer`</u>. Pressure is the normal component of the stress, $p = -\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})$.

292. The relation between shear and stress?

<u>answer</u>. Shear is the tangential component of the stress

293. What do the 9 components of a stress tensor describe?

<u>answer</u>. They describe the stresses (force/area) on three sides of an infinitesimal cube.

2019-06-21

294. Is S^n, $n \geq 1$ contractible?

<u>answer</u>. NO!

295. Is S^n, $n \geq 1$ simply-connected?

<u>answer</u>. Yes if $n \geq 2$. S^1 is not simply-connected.

2019-06-18

296. Let $f : M \to \mathbb{R}$ where M is n-dimensional. Find the Morse index of f at a nondegenerate max, min and saddle.

<u>answer</u>.

$$f \circ y^{-1}(y_1, y_2, \ldots, y_n) = f(p) - y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdot\cdot y_{k+2}^2 + \cdots + y_n^2$$

The integer k, the number of negative signs in the quadratic form, is the Morse index of the critical point.

The Morse index at p is n if p is an argmax, 0 if p is an argmin and between 0 and n if $f(p)$ is a saddle.

297. The geostrophic approximation is a simplification of the equations governing

<u>answer</u>. the horizontal components of velocity.

298. The geostrophic approximation is valid when the largest terms in the equations of motion are

<u>answer</u>. those involving the Coriolis force and the pressure gradient.

299. Spatial and temporal physical situations where the assumption of geostrophic approximation are generally true?

<u>answer</u>. In the deep ocean over large (over 100 km) spatial and long (over 2 days) temporal scales.

300. Obtain the geostrophic equation.

answer. For frictionless motion, the horizontal components of the momentum equation are

$$\frac{\mathrm{D}\mathbf{u}_{\mathrm{H}}}{\mathrm{Dt}} + \mathbf{f}_H = -\frac{1}{\rho}\nabla_H \mathrm{p}$$

where $\mathbf{f}_H$ is the horizontal component of the Coriolis force $2\boldsymbol{\Omega} \times \mathbf{u}$. In the limit $Ro \to 0$,

$$\mathbf{f}_H = -\frac{1}{\rho}\nabla p$$

This is called geostrophic equation.

In the GFD case, the vertical component of velocity and Coriolis acceleration is negligible and the horizontal Coriolis acceleration becomes

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{f}_H = \begin{pmatrix} -v \\ u \end{pmatrix} f$$

where $f = 2\Omega \sin\varphi$.

In open form, the equations read

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} = -fu$$

301. The equations of the geostrophic approximation are?

answer.

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} = -fu$$

where $f = 2\Omega \sin\theta$ is the Coriolis parameter, the component of the earth's rotation perpendicular to the ocean surface.

302. The number which describes how good the geostrophic approximation is?

answer. If the Rossby number is small then the geostrophic approximation is good.

303. For which range of Rossby numbers is the geostrophic approximation good?

answer. If Rossby number is smaller than 0.1 then it is usually OK to use geostrophic approximation.

304. What is the equation of Rossby number?

answer. $\mathrm{Ro} = \frac{U}{fL}$.

305. Describe the Rossby number as the ratio of two forces?

answer. It is the ratio of the inertial force ($|\mathbf{v}\cdot\nabla\mathbf{v}| \sim U^2/L$) to the Coriolis force ($\Omega \times \mathbf{v} \sim U\Omega$)

306. What is inertial force in NSE?

answer. $\rho \mathbf{u} \cdot \nabla \mathbf{u}$ term. It is due to the momentum of the fluid.

The first component of the inertial force F_1 measures how much u_1 changes in the direction of $\mathbf{u}$. If u_1 stays constant following the flow, then F_1 is zero. The inertial force is non-zero if and only if the velocity field is changing direction along the flow, that is the flow is not linear.

307. In small Rossby numbers, which term is dominant?

answer. Coriolis forces.

308. In large Rossby numbers, which term is dominant?

answer. Inertial and centrifugal forces.

309. What is Rossby number in tornadoes? What is the conclusion?

answer. Large, in the order of 1000. Coriolis force is negligible and balance is between pressure and centrifugal forces (called cyclostrophic balance)

310. In a bathtub, what is Rossby number? What is the conclusion?

answer. Length scale is small so Rossby number is large. Planetary rotation is unimportant.

311. In tropics and lower latitudes, what is Rossby number? What is the conclusion?

answer. Rossby number is large since the Coriolis parameter f is small. Planetary rotation is unimportant.

312. What is hydrostatic approximation for the ocean circulation model? When is it valid?

answer. The hydrostatic approximation is a simplification of the equation governing the vertical component of velocity.

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \mathcal{F}_z$$

We can see that if friction and the vertical acceleration Dw/Dt are negligible compared to gravity, we obtain: $\frac{\partial p}{\partial z} = -\rho g$.

That is pressure depends only on the depth.

313. What is the most realistic set of equations for modelling the ocean circulation? What are its variables?

answer. Primitive equations with variables 3D velocity, density (as a function of temperature, salinity, and pressure), temperature, salinity, and pressure (due both to variations in the sea level and the internal density field).

Since it takes a tremendous amount of computer power and time to solve the full set of equations, a number of different approximations are often used to simplify the equations to make them quicker and easier to solve.

314. How can dissipative systems maintain persistent behaviours?

 answer. Since they lose energy in time they must have influx of energy/matter.

2019-06-16

315. What is a Rossby wave? Under which conditions they occur?

 answer. Rossby waves, also known as planetary waves, naturally occur in rotating fluids. Within the Earth's ocean and atmosphere, these waves form as a result of the rotation of the planet. These waves affect the planet's weather and climate.

316. Let M and N be smooth manifolds and $f : M \to N$ a smooth map. Show that f has a pullback $f^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$.

 answer. We need to show that f^* sends closed forms on N to closed forms on M and exact forms on N to exact forms on M.

 Let θ be a closed form on N then $0 = f^* d\theta = df^*\theta$ since $d\theta = 0$.

 Now let ω be an exact form, thus $\omega = d\psi$. Then $f^*\omega = f^* d\psi = df^*\psi$ and $f^*\omega$ is exact.

317. Define pullback of smooth maps.

 answer. Just precomposition. $f : M \to N$, $g : N \to A$ then $f^* g = g \circ f : M \to A$.

318. For $f : M \to N$, define the pullback $f^*(\omega)$ of a k-form ω on N.

 answer. $f^*(\omega)(X_1, \ldots, X_k) = \omega(df(X_1), \ldots, df(X_k))$ where X is a tangent vector on M.

 The pullback is defined via pushforward.

319. Show that pullback and exterior derivative commutes.

 answer. ToDO

320. What is $H^k_{\mathrm{dR}}(M)$, $k \geq 0$ if M is a contractible manifold?

 answer. $H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$

321. Enthalpy form of the first law of thermodynamics?

 answer. $dh = Tds + vdp$, where dh denotes the enthalpy change, T the temperature, ds the change in entropy, $v = \frac{1}{\rho}$ the specific volume, and p the pressure.

322. Effect of adiabatic process on the enthalpy form of the first law of thermodynamics?

answer. Put $ds = 0$ in $dh = Tds + vdp$ where $v = 1/\rho$. h is enthalpy.

323. First law of thermodynamics?

answer. Energy is conversed.

324. What is specific volume?

answer. 1/density or V/m.

2019-06-14

325. What are the properties of material derivative?

answer. Linear $\frac{D(af+bg)}{Dt} = a\frac{Df}{Dt} + b\frac{Dg}{Dt}$ and satisfies Leibniz (product) rule $\frac{D(fg)}{Dt} = \frac{Df}{Dt}g + f\frac{Dg}{Dt}$.

326. What is the relation between stratification of flows and gravity? Are stratified flows common?

answer. Gravity causes heavier fluids to sink down. The Earth's oceans and atmosphere are both stratified, and so such stratified flows are extremely common.

327. Does a volume-preserving system on a domain $D \subset \mathbb{R}^n$ always have a first integral?

answer. If every closed $n-1$ form on D is exact. This is true if and only if the (n-1)th de-Rham cohomology vanishes. This in turn holds if the space contractible.

When $n = 2$, this is true

328. Recall Taylor's theorem in integral form.

answer. Taylor's Theorem with integral form is explicit, does not involve an unknown c.

$$f(a+\Delta x) = f(a)+f'(a)\Delta x+\frac{f''(a)}{2!}(\Delta x)^2+\cdots+\frac{f^{(n)}(a)}{n!}(\Delta x)^n+R_{n,a}(\Delta x)$$

where $R_{n,a}(\Delta x) = \frac{(\Delta x)^{n+1}}{n!}\int_0^1 f^{(n+1)}(a + t\Delta x)(1 - t)^n dt$.

329. (Morse Lemma. One variable case) Consider the Taylor expansion at a nondegenerate critical point $f(a + x) - f(a) = h(x)x^2$, i.e. $f'(a) = 0$,

$f''(a) \neq 0$. Here $h(x) = \int_0^1 f''(a+tx)(1-t)dt$. Make a change of coordinates $y = p(x)$, $p(0) = 0$ so that

$$\Delta f(x) = f(a+x) - f(x) = ky^2, \qquad k = \begin{cases} +1, & f''(a) > 0 \\ -1, & f''(a) < 0 \end{cases}$$

answer. Let $y = p(x) = x\sqrt{|h(x)|}$. Then

$$y^2 = x^2|h(x)| = x^2 h(x) \begin{cases} +1, & h(x) > 0 \\ -1, & h(x) < 0 \end{cases}$$

Check that $h(0) = f''(a) \int_0^1 (1-t)dt = \frac{f''(a)}{2}$. So $h(x) > 0$ (resp. $<$) in a small neighborhood if $f''(a) > 0$ (resp. $<$).

Now check that $y = p(x)$ is a coordinate transformation, i.e. $p'(0) \neq 0$.

$$p'(x) = \sqrt{|h(x)|} \pm \frac{h'(x)}{2\sqrt{|h(x)|}} x, \qquad \text{if } h(x) \neq 0$$

Hence $p'(0) = \sqrt{|h(0)|} = \sqrt{|f''(a)|/2} \neq 0$

2019-06-13

330. What is a homology sphere?

answer. A homology sphere is an n -manifold X having the homology groups of an $n-$ sphere, for some integer $n \geq 1$. That is, $H_0(\mathrm{X}, \mathrm{Z}) = \mathrm{Z} = \mathrm{H}_n(\mathrm{X}, \mathrm{Z})$ and $H_i(X, Z) = \{0\}$ for all other i. Therefore X is a connected space, with one non-zero higher Betti number: b_n. It does not follow that X is simply-connected, only that its fundamental group is perfect (see Hurewicz theorem).

331. In plane in which domains is $d\theta$ non-exact?

answer. In a domain which does encircle the origin.

332. Define the kth-Betti number of a smooth manifold M.

answer.

$$b_k(M) := \dim H^k_{\mathrm{dR}}(M) = \dim \frac{\text{closed k-forms}}{\text{exact k-forms}}.$$

333. What is the zeroth Betti number?

answer. $b_0(M) = \dim H^0_{\mathrm{dR}}(M)$ is the number of connected components of M

334. Let M and N be homotopy equivalent manifolds, $M \simeq N$. What can be said about their de-Rham cohomologies?

answer. They are isomorphic at each level, i.e. $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$ for all $k \in \mathbb{N}$.

Why? TodO.

335. Which properties do homotopy equivalance preserve?

answer. X is path-connected if and only if Y is. X is simply-connected if and only if Y is.

336. What is the relation between $\mathbb{R}^n - \{0\}$ and S^{n-1}?

answer. There is a homotopy equivalence between them.

337. Show that $\mathbb{R}^n - \{0\}$ and S^{n-1} are homotopy equivalent.

answer.

$$H(x,t) = t\frac{x}{|x|} + (1-t)x, \qquad H(\cdot,0) = \mathrm{id}_{\mathbb{R}^n - \{0\}}, \qquad H(\cdot,1)\,|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$$

H is a deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1} and as a result two spaces are homotopy equivalent.

338. Homotopy and contractibility?

answer. Spaces that are homotopy equivalent to a point are called contractible. Intuitively, two spaces X and Y are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations.

339. Is it true that every homeomorphism is a homotopy equivalence? Prove it or give a counterexample.

answer. It is true. If $f : X \to Y$ is a homeomorphism then let $g = f^{-1}$ so that $f \circ g = id_Y$ and hence trivially homotopic to id_Y. Conversely, $g \circ f = id_X$.

340. Suppose $Y \subset X$, $f : X \to Y$. If $H(x,0) = id_X$ and $H(x,1) = f(x)$ then are X and Y homotopy equivalent?

answer. Let $g : Y \to X$ such that $g(y) = y$, $\forall y \in Y$. Suppose that $H(x$

341. Define the homotopy equivalance (the same homotopy type) of two spaces X and Y.

answer. There exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_x and $f \circ g$ is homotopic to id_y.

That is there exists homotopies $H : X \times I \to X$ and $G : Y \times I \to Y$ such that

$$H(x, 0) = x = \mathrm{id}(x), \quad H(x, 1) = g \circ f(x)$$

$$G(y, 0) = y = \mathrm{id}(y), \quad G(y, 1) = f \circ g(y)$$

342. Let X be any topological space and A be a convex subset of $\mathbb{R}^n$. Then any two continuous maps $f, g : X \to A$ are homotopic.

<u>answer</u>. Take $F(x, t) = (1 - t)f(x) + tg(x)$

343. Let $f, g : \mathbb{R} \to \mathbb{R}$ any two continuous, real functions. Then $f \simeq g$, i.e. homotopic.

<u>answer</u>. Take $F(x, t) = (1 - t) \cdot f(x) + t \cdot g(x)$.

2019-06-12

344. Give an example of a linear unstable ODE system with spectrum $Re\lambda \leq 0$.

<u>answer</u>. $\dot{x} = Ax$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Then $y = y_0$, $x = x_0 + y_0 t$.

345. When are the frictional effects negligible for the oceanic/atmospheric flows?

<u>answer</u>. For typical atmospheric and oceanic flows, frictional effects are negligible except close to boundaries where the fluid rubs over the Earth's surface.

346. How deep is the atmospheric boundary layer?

<u>answer</u>. The atmospheric boundary layer – which is typically a few hundred meters to 1 km or so deep

347. The atmospheric boundary layer is complicated because of (1) and (2).

<u>answer</u>. (1) the surface is not smooth, (2) the boundary layer is usually turbulent.

Extra: (1) there are mountains, trees, and other irregularities that increase the exchange of momentum between the air and the ground. (This is the main reason why frictional effects are greater over land than over ocean). (2) contains many small scale and often vigorous eddies; these eddies can act somewhat like mobile molecules, and diffuse momentum more effectively than molecular viscosity. The same can be said of oceanic boundary layers which are subject, for example, to the stirring by eddies generated by the action of the wind.

348. Derive the x component of the pressure force acting on a cubic fluid parcel within the fluid.

<u>answer</u>. Take a cubic fluid parcel with center at (x, y, z) and sides δx, δy, δz. The net x -component of the pressure force is

$$F_x = \left[p\left(x - \frac{\delta x}{2}, y, z\right) - p\left(x + \frac{\delta x}{2}, y, z\right)\right] \delta y \delta z$$

where the first term is the inward pressure on the left face, and the second one on the right face. By Mean Value Theorem, x-component of the pressure force is

$$F_x = -\frac{\partial p}{\partial x} \delta x \delta y \delta z$$

2019-06-11

349. Write the product rule $\frac{D(fg)}{Dt}$.

<u>answer</u>. $g\frac{Df}{Dt} + f\frac{Dg}{Dt}$.

350. Is an incompressible flow barotropic or baroclinic? What about homogeneous flow?

<u>answer</u>. Can be both. Barotropic/baroclinic assumption is about equation of state. However a homogeneous flow is barotropic since density is a (constant) function of pressure.

351. Gradient of a radial function $f(|\mathbf{r}|)$?

<u>answer</u>. $f'(|\mathbf{r}|)\frac{\mathbf{r}}{|\mathbf{r}|} = f'(|\mathbf{r}|)\widehat{\mathbf{r}}$.

For a radial function, the level surfaces are spheres and the gradient field points in the radial direction which is normal to the spheres.

352. In general barotropic means compressible.

353. Let $p : \mathbb{R} \to \mathbb{R}$, $\rho : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be smooth. Show that $f(\rho)dp(\rho)$ is exact.

<u>answer</u>. Let $w = \int^{\rho} f(\lambda)p'(\lambda)d\lambda$. Then $dw = f(\rho)p'(\rho)d\rho = f(\rho)dp(\rho)$.

2019-06-10

354. For a vector $\mathbf{A}$ of fixed length, rotating about the origin with constant angular velocity $\mathbf{\Omega}$ DERIVE $\frac{d\mathbf{A}}{dt} = \mathbf{\Omega} \times \mathbf{A}$.

<u>answer</u>. Let γ be the angle between $\mathbf{\Omega}$ and $\mathbf{A}$.

$$d\mathbf{A} = \mathbf{A}(t + dt) - \mathbf{A}(t) = \mathbf{e}|\mathbf{A}| \sin\gamma d\theta = \mathbf{e}|\mathbf{A}| \sin\gamma |\mathbf{\Omega}| dt$$

where $\mathbf{e}$ is the unit-vector along $d\mathbf{A}$. Hence $\mathbf{e} \perp \mathbf{A}$ and $\mathbf{e} \perp \mathbf{\Omega}$ so that $\mathbf{e} = \frac{\mathbf{\Omega}\times\mathbf{A}}{|\mathbf{\Omega}\times\mathbf{A}|} = \frac{\mathbf{\Omega}\times\mathbf{A}}{|\mathbf{\Omega}||\mathbf{A}| \sin\gamma}$.

355. For a vector $\mathbf{A}$ of fixed length, rotating about the origin with constant angular velocity $\boldsymbol{\omega}$, $\frac{d\mathbf{A}}{dt} =?$

answer. $\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}$.

356. GFD Momentum equations in a rotating frame?

answer. $\rho\left[\frac{d\mathbf{u}}{dt} + 2\boldsymbol{\omega} \times \mathbf{u}\right] = -\nabla p + \rho\nabla\Phi + F$ Pedlosky 1.6.7

357. Give an integral condition for a 1-form ω to be exact on M?

answer. ω is exact if and only if $\int_\gamma \omega = 0$ for all closed curves $\gamma \subset X$. This is equivalent to saying that $\int_A^B \omega$ doesn't depend on path (for any $A, B \in X$). Hence $\alpha = \int_A^x \omega$ can be defined for which $d\alpha = \omega$.

358. Give an integral condition for a k-form ω to be exact on M?

answer. ω is exact if and only if it is conservative that is $\int_\gamma \omega = 0$ for all orientable closed k submanifolds of M. By Stoke's Theorem these definitions are equivalent.

359. How to keep in mind the sign of the pressure in NSE?

answer. Acceleration is produced by negative pressure gradient since the flow is from high pressure to low pressure.

2019-06-09

360. If X is a vector field and $f : M \to \mathbb{R}$ then $X(f) =?$ Why?

answer. $df(X)$. To see, let $X = X_i\partial_i$ and $df = \partial_i f dx^i$.

$$X(f) = X_i\partial_i f = df(X)$$

361. Are biological systems conservative or dissipative?

answer. Dissipative.

362. What is the volumetric part of the Cauchy stress tensor?

answer. $-p\mathbf{I}$ where $p = \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})$ is the mechanical pressure.

363. For which stress tensor does the NSE reduce to Euler's equation?

answer. When the stress tensor is $\mathbf{T} = -p\mathbf{I}$.

364. Describe the isotropic/anisotropic decomposition of a tensor field. Are they Symmetric/antisymmetric?

answer. Given a tensor T, isotropic part $T_{iso} = \frac{1}{3}\operatorname{tr}(T)I$ is the same as the isotropic part of the symmetric part $S = \frac{1}{2}\left(T + T^T\right)$ since $\operatorname{tr}(S) = \operatorname{tr}(T)$. The anisotropic part of T is $T_{an} = T - T_{iso}$ is traceless but not neccessarily symmetric/anti-symmetric.

365. Volumetric/deviatoric decomposition of a tensor? What do they describe?

`answer`.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{vol} + \boldsymbol{\sigma}_{dev} = \left(\frac{1}{3}\,\mathrm{tr}\,(\boldsymbol{\sigma})\,\mathbf{I}\right) + \left(\boldsymbol{\sigma} - \frac{1}{3}\,\mathrm{tr}\,(\boldsymbol{\sigma})\,\mathbf{I}\right)$$

366. What do the volumetric/deviatoric decomposition terms of a tensor describe?

`answer`. Volumetric (aka hydrostatic, mean, isotropic) part is traceful and is related to the volume change, deviatoric part is traceless and is related to the shape change.

367. How does the stress tensor $\boldsymbol{\sigma}$ and external force $\mathbf{f}$ show up in Navier Stokes?

`answer`.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}$$

368. Write the Navier Stokes equation when the stress tensor consists only shear part.

`answer`. It becomes Euler equations.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} = -\nabla p + \mathbf{f}.$$

369. Given $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\hat{\mathbf{s}}}{s^2}\rho\left(\mathbf{r}'\right)d\tau'$ where $\mathbf{s} = \mathbf{r} - \mathbf{r}'$, COMPUTE the divergence of $E(\mathbf{r})$. (This is Gauss's law for electricity which is the first law of Maxwell.)

`answer`. $\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho(\mathbf{r})$ found by using $\nabla\cdot\left(\frac{\hat{\mathbf{s}}}{s}\right) = 4\pi\delta^3(\mathbf{s})$.

370. Gauss's Law in integral form?

`answer`. $\oint_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\int_{\mathcal{V}}\rho d\tau$.

371. Gauss Law for electrostatic field in differential form?

`answer`. $\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$.

372. Show directly that the flux of electric field through a sphere of radius r of a single charge q located at the origin is $\frac{q}{\epsilon_0}$.

`answer`. $\oint_S \mathbf{E}\cdot d\mathbf{A} = \oint_S \frac{1}{4\pi\epsilon_0}\left(\frac{q\hat{\mathbf{r}}}{r^2}\cdot r^2\sin\theta d\theta d\phi\hat{\mathbf{r}}\right) = \frac{1}{\epsilon_0}q$.

373. The flux of electric field through a closed surface?

`answer`. $\oint_S \mathbf{E}\cdot d\mathbf{A} = Q_{enc}/\epsilon_0$ where Q_{enc} is the total charged enclosed, and ϵ_0 is the permittivity of the vacuum. This is called Gauss Law.

2019-06-08

374. What is the electrostatic Field at $\mathbf{r}$ due to a continuous charge distribution in a volume V?

answer. $\mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}')d\mathbf{r}'$ where ρ is the charge density.

This can be derived from $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$.

This can be generalized to electric fields due to line charges and surface charges.

375. Electrostatic Field at $\mathbf{r}$ due to n charges q_i located at $\mathbf{r}_i$?

answer. $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^n q_i \dfrac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}$.

376. Electrostatic field at $\mathbf{r}$ due to a single charge q located at $\mathbf{r}_0$?

answer. $\mathbf{E} = \frac{1}{4\pi\epsilon_0} q \dfrac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}$.

377. What does the electric field at a point describe?

answer. The electric field at a point is the net Coloumb's force at that point due to charges at the other points.

378. (Coloumb's Law) Electrostatic force between two charged particles?

answer. The force of charge q to charge Q located at $\mathbf{r}$ relative to q. $\mathbf{F} = \frac{1}{4\pi\epsilon_0} qQ \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$.

379. A k-form on D is exact if its integral over any compact oriented k-manifold in D is zero.

answer. Is this true?

2019-06-05

380. What is an equation of state?

answer. A relation between state variables. For ocean: $f(\rho, P, S, T) = 0$

381. The curl of vorticity of a divergence-free vector field acts as negative diffusion. Why?

answer. $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u} = -\nabla \times \omega$.

382. Define a barotropic fluid as an equation of state.

answer. A barotropic fluid has an equation of state $p = p(\rho)$ or $\rho = \rho(p)$ which does not contain the temperature as a dependent variable. Thus level surfaces of density and pressure coincide.

383. Define a baroclinic fluid.

`answer`. The flow is baroclinic if it is not barotropic. That is density is a not just a function of only pressure but is a function of both pressure and temperature.

2019-06-04

384. Every closed 2-form is exact in a domain in D if any closed surface S in D is the boundary of a region V in D.

I think this means, the region can not contain any holes.

385. Show that a simply-connected domain is homologically trivial, that is any curve C in D is the boundary of a surface in D.

`answer`. Let $c(t)$ be a parametrization of the curve C and $H(t, 0) = c(t)$ and $H(t, 1) = p$ be a homotopy between the curve and a point. Then $H(t, s)$ is a parametrization of a surface D with boundary C. This is difficult to prove?

386. A domain D for which any closed curve $c \subset D$ is the boundary of a surface $S \subset D$ is called homologically trivial. A closed 1-form is exact in a homologically trivial domain. Why?

`answer`. By Stokes Theorem, the 1-form ω is path-independent. We can define a 0-form $\alpha(p) = \int_{p_0}^{p} \omega$ so that $d\alpha = \omega$.

387. Give an example. A linear stable ODE system with spectrum not contained in the complex left half plane.

`answer`. Simple pendulum. $\ddot{x} + x = 0$.

388. When is a linear system having eigenvalues of zero real part stable/unstable?

`answer`. Stable if algebraic and geometric multiplicities of ALL eigenvalues with zero real part are equal. Unstable otherwise.

389. $\dot{J} = (\nabla \cdot \mathbf{u})J$ implies $J(t) = J_0 \exp\left(\int \nabla \cdot \mathbf{u}\right)$. Does this mean anything?

390. Find $\frac{d}{dt} Vol(\Omega(t))$.

`answer`.

$$\begin{aligned}\frac{d}{dt} Vol(\Omega(t)) &= \frac{d}{dt} \int_{\Omega(t)} 1 dV(x) = \frac{d}{dt} \int_{\Omega_0} J dV(a) = \int_{\Omega_0} \dot{J} dV(a) \\ &= \int_{\Omega_0} (\nabla \cdot \mathbf{u}) J dV(a) = \int_{\Omega(t)} (\nabla \cdot \mathbf{u}) dV(x)\end{aligned}$$

391. If every closed k-form is exact, what can be said about the de-Rham cohomology?

answer. k-th de-Rham cohomology vanishes.

2019-06-01

392. When does the pullback $f(x,t) = f_0(\phi^{-1}(x,t))$ imply that f is transported by $\mathbf{u}$?

answer. When u is divergence-free. To see, given condition implies $f_t + \mathbf{u} \cdot \nabla f = 0$ which in integral form does not imply $\frac{d}{dt}\int_V f dV + \int_{\partial V} f \mathbf{u} \cdot \mathbf{n} = 0$ unless u is divergence-free.

393. Let ϕ be the flow of an incompressible velocity field $\mathbf{u}$ and suppose f is transported by $\mathbf{u}$. If f_0 is a probability density function then so is $f(\cdot,t)$ for all t and $P_t(\Omega_t) = P(\Omega_0)$.

answer. We have $J(x,t)f(x,t)dV(x) = J(a,0)f_0(a)dV(a)$ and $J \equiv= 1$. By the change of variables formula

$$\int_{\Omega_t} f(x,t)dV(x) = \int_{\Omega_0} J(a,t)f(a,0)dV(a).$$

394. Consider the motion ϕ_t generated by $\mathbf{u}$ and let $\Omega_t = \phi_t(\Omega_0)$. When is $\int_{\Omega_t} f(x,t)dV(x) = \int_{\Omega_0} f(a,0)dV(a)$?

answer. In the incompressible case, that is $J(a,t) = \det(\nabla_a \phi_t) \equiv 1$ since

$$\int_{\Omega_t} f(x,t)dV(x) = \int_{\Omega_0} J(a,t)f(a,0)dV(a)$$

by the change of variables.

395. Show that if the infinitesimal volumes are conserved (i.e. divergence of flow is zero) then global volumes are also conserved.

answer. Let ϕ be the flow of an incompressible velocity field $\mathbf{u}$ and suppose f is transported by $\mathbf{u}$, that is define $f(x,t) = f_0(\phi^{-1}(x,t))$. If f_0 is a probability density function then so is $f(\cdot,t)$ for all t and $P(\Omega_t) = P(\Omega_0)$.

answer. ToDO: transport http://faculty.virginia.edu/rohde/transport/OTCrashCourse.pdf.

This is a consequence of the change of variables formula

$$\int_{\Omega_t} f(x,t)dV(x) = \int_{\Omega_0} J(a,t)f(a,0)dV(a).$$

396. Let $J(x,t)f(x,t) = f_0(a)$ where $J > 0$ is the Jacobian of the motion ϕ generated by $\mathbf{u}$ and $x = \phi(a,t)$. Find the equation satisfied by f. What happens in the special case $J \equiv 1$?

answer.

$$\frac{D\,(Jf)}{Dt} := f_t + \nabla \cdot (f\mathbf{u}) = 0$$

$$J \equiv 1 \implies \frac{Df}{Dt} := f_t + \mathbf{u} \cdot \nabla f = 0$$

To see, Take the time derivative of both sides and use $DJ/Dt = J(\nabla \cdot u)$ to get

$$0 = \frac{DJ}{Dt} f + J(f_t + \mathbf{u} \cdot \nabla f) = J(f(\nabla \cdot \mathbf{u}) + f_t + \mathbf{u} \cdot \nabla f) = f_t + \nabla \cdot (f\mathbf{u})$$

397. Research statistical mechanics for ODEs.

answer. ToDO

398. If the flow map is volume preserving in the phase space then if an ensemble of initial data in the phase space is stretched in some direction under the flow map, it must be compensated for by squeezing in some other direction under the flow map and vice versa. This kind of stretching/squeezing together with the bending/twisting (mixing) implied by the non-linearity is a key mechanism in the complex/chaotic behavior of the dynamical system and in the tendency towards statistical equilibrium for large ensembles of trajectories.

2019-05-31

399. What are the characteristics of GFD?

answer. 1) shallow or thin, i.e., horizontal length scales is much larger than the vertical length scales, 2) stratified, i.e., the vertical variation in density is important, and 3) rapidly rotating.

400. (Reynolds' Transport Theorem)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \theta \mathrm{d}V = \int_{V(t)} \left(\frac{\partial \theta}{\partial t} + \mathrm{div}(\theta \mathbf{u}) \right) \mathrm{d}V$$

What do the terms on the right describe?

answer. First term is generated by the unsteadiness of the field.

Second term is the transport of θ by $\mathbf{u}$ across the boundary of $V(t)$.

401. What is the type of matrix whose columns form an orthonormal basis in $\mathbb{C}^n$?

answer. Unitary. $AA^* = I$.

402. For which $\mathbf{A}$, $\iint_S \mathbf{A} \cdot d\mathbf{S} = 0$ for every orientable surface S without boundary?

answer. This is the definition of $\mathbf{A} \cdot d\mathbf{S}$ being an exact 2-form, that is $\mathbf{A} = \nabla \times \mathbf{F}$. Then by Stokes Theorem $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$.

Alternatively, if $\mathbf{A}$ is closed ($\nabla \cdot \mathbf{A} = 0$) and S is the boundary of a volume region V then

$$\iint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{A} dV = 0$$

403. Why is a contractible space path-connected?

answer. The identity map on X is homotopic to the constant map $x \to x_0$, that is $H(0, x) = x$, $H(1, x) = x_0$ and H is continuous. Take any point $p \in X$ then $c(t) = H(t, p)$ is a path between p and x_0. Hence any point p is path connected to x_0.

404. Two layer topography https://www.duo.uio.no/bitstream/handle/10852/64575/jostein_brandshoi_master_2018.pdf?sequence=1&isAllowed=y

405. Shallow water equations? http://www.mathematik.tu-dortmund.de/lsiii/cms/papers/Kuehbacher2009.pdf http://pordlabs.ucsd.edu/jen/gfd/hw/shallow_water_pv.pdf http://www.cgd.ucar.edu/staff/islas/teaching/5_Shallow_QG.pdf http://barnes.atmos.colostate.edu/COURSES/AT601_F15/lecture_material/12_lecture_notes_handouts.pdf https://gfd.whoi.edu/wp-content/uploads/sites/18/2018/03/lecture8-harvey_136564.pdf

406. Basic tensor calculus?

407. 3D vorticity equation for a baroclinic, compressible, inviscid flow?

answer. $\frac{D\vec{\omega}}{Dt} \equiv \frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{\omega} = \underbrace{(\vec{\omega} \cdot \vec{\nabla})\vec{u}}_{\text{vortex stretching}} - \vec{\omega}(\vec{\nabla} \cdot \vec{u}) + \underbrace{\frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p}_{\text{baroclinic contribution}}$.

2019-05-30

408. When does an incompressible irrotational flow on a given domain exist and is unique?

answer. If the domain D is simply-connected and either (1) $\mathbf{u} \cdot \mathbf{n}$ or (2) $\mathbf{u} \times \mathbf{n}$ is specified.

In either case the flow MUST be given by the gradient of a harmonic potential, $\mathbf{u} = \nabla\phi$. There are infinitely many harmonic functions unless a boundary condition is specified.

(1) $\mathbf{u} \cdot \mathbf{n} = g$ on ∂D implies $\nabla\phi \cdot \mathbf{n} = g$ on ∂D. Hence by the Neumann problem, vector potential is unique up to a constant which means $\mathbf{u}$ is unique.

(2) $\mathbf{u} \times \mathbf{n} = \mathbf{g}$ on ∂D implies $\nabla\phi \times \mathbf{n} = \mathbf{g}$ on ∂D. ??? When $\mathbf{g} = 0$ and in 2D this means ϕ is constant on the boundary curve which is the Dirichlet condition for the potential.

Q. What happens on multiply-connected domains?

409. (Baroclinic term) $-\oint_C \frac{\nabla p}{\rho} \cdot d\vec{x} = \iint_A ?$

<u>answer</u>. $\displaystyle\int_A \frac{\nabla\rho \times \nabla p}{\rho^2} \cdot \hat{n} dA.$

410. Basic Stokes Theorem and meaning?

<u>answer</u>. $\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA = \oint_{\partial A} \mathbf{u} \cdot d\mathbf{r}$. Sum of small circulations equal to big circulation.

411. What is baroclinity vector? What does it measure?

<u>answer</u>. $\nabla p \times \nabla\rho$ is called the baroclinity vector and measures whether constant density (isopycnic) and constant pressure (isobar) surfaces are aligned. A barotropic fluid is one for which baroclinity vector is zero.

412. State Kelvin's Circulation Theorem.

<u>answer</u>. For

$$\frac{D\mathbf{u}}{Dt} = \nabla\Phi$$

$$\frac{d}{dt}\int_{\Omega_t} \mathbf{u} \cdot d\mathbf{x} = 0$$

If the material derivative of the velocity is irrotational then the circulation of velocity around a material curve is a conserved property.

For example inviscid barotropic flow under conservative forces.

413. What is the relation between the $\nabla^2\mathbf{u}$ and the vorticity for a divergence free field?

<u>answer</u>. $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u} = -\nabla \times \omega$.

414. What is $\frac{d\Gamma}{dt}$, the rate of circulation for viscous flows?

<u>answer</u>. $\displaystyle\frac{d\Gamma}{dt} = \oint_{C_t} (-\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}) \cdot d\mathbf{x}.$

To see,

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_{C_t} \mathbf{u}\cdot d\mathbf{x} = \oint_{C_t} \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{x} + \oint_{C_t} \mathbf{u}\cdot d\mathbf{u} = \oint_{C_t} \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{x}$$

415. Baroclinity and viscous forces gives rise to circulation. Why?

`answer`. $\frac{d\Gamma}{dt} = -\oint_{C_t}(\frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u})\cdot d\mathbf{x}$.

416. Under which boundary conditions is a harmonic function unique?

`answer`. ToDO

417. Up to a choice of an orthonormal basis, a normal matrix is ...

`answer`. a diagonal matrix.

418. Necessary and sufficient conditions for a matrix to have a set of eigenvectors which form an orthonormal basis?

`answer`. If and only if it is normal. Easy to remember: The eigenbasis is orthonormal iff the matrix is normal. Proof in other questions.

419. Show that if a matrix is normal then it has a set of eigenvectors which form an orthonormal basis.

`answer`. Suppose A is normal then by Schur decomposition $A = UTU^*$ where U is unitary and T is triangular. Since $AA^* = A^*A$,

$$UTT^*U^* = AA^* = A^*A = UT^*TU^*$$

$TT^* = TT^*$ and T is normal as well. A normal triangular matrix must be diagonal. Thus T is a diagonal matrix of eigenvalues of A, U is the unitary matrix of eigenvectors.

420. If $A \in M_n(\mathbb{C})$ has an orthonormal basis then show that A is normal.

`answer`. Suppose U is a unitary matrix of eigenvectors of A then $u_j^*Au_i = u_j^*\lambda_i u_i = \lambda_i\delta_{ij}$, that is $U^*AU = D$ or $A = UDU^*$. This shows that

$$AA^* = (UDU^*)(UD^*U^*) = UDD^*U^* = UD^*DU^* = A^*A.$$

421. If $\{u_1,\ldots,u_n\}$ form an orthonormal basis for $\mathbb{C}^n$ then what is the property of the matrix $U = (u_1,\ldots,u_n)$? Why?

`answer`. U is unitary since $(u_i,u_j) = u_i^*u_j = \overline{u_i}^T u_j = \delta_{ij}$.

422. Express $\mathbf{u}\cdot\nabla\mathbf{u}$ as a dyadic?

`answer`.

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}(\mathbf{u})\mathbf{u}$$

since

$$\left(\operatorname{div}(\mathbf{u}\otimes\mathbf{u})\right)_j = \partial_i\left(\mathbf{u}_i\mathbf{u}_j\right) = \operatorname{div}(\mathbf{u})\mathbf{u}_j + \mathbf{u}_i\mathbf{u}_{j,i}.$$

423. What is the dyadic product $\mathbf{u} \otimes \mathbf{v}$?

answer. The tensor $\mathbf{u}_i \mathbf{v}_j$.

2019-05-29

424. Euler's equation including gravity?

answer. $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - g\mathbf{k}$

425. What is the hydrostatic balance equation for a homogeneous fluid at rest?

answer. $\frac{dp}{dz} = -\rho(z)g$ or $p = -g \int_0^z \rho(z')dz'$.

426. Derive the hydrostatic balance equation $\frac{dp}{dz} = -\rho g$ for a homogeneous fluid at rest using the vertical balance between pressure and gravity. (This balance can also be derived from NSE)

answer. Fluid parcel at equilibrium. $F_{top} = -p_{top}A$, surface area A. $F_{bottom} = p_{bottom}A$. The weight of the parcel $F_{weight} = -\rho A \Delta z g$. Balancing $F_{top} + F_{bottom} + F_{weight} = 0$ gives $\Delta p = -\rho g \Delta z$.

427. What does the principle of hydrostatic equilibrium tell?

answer. The principle of hydrostatic equilibrium is that the pressure at any point in a fluid at rest (whence, hydrostatic) is just due to the weight of the overlying fluid.

2019-05-28

428. Surface element in constant radius r in spherical coordinates:

answer. $dS = ds_\theta ds_\phi = r^2 \sin\theta d\theta d\phi$.

429. The volume element in spherical coordinates?

answer. $dv = ds_r ds_\theta ds_\phi = r^2 \sin\theta dr d\theta d\phi$.

430. The line element in spherical coordinates?

answer. $d\mathbf{l} = ds_r \hat{\mathbf{r}} + ds_\theta \hat{\boldsymbol{\theta}} + ds_\phi \hat{\boldsymbol{\phi}} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$.

431. What is the circulation of a gradient field?

answer. Zero.

2019-05-27

432. Prove that $KerA^T = Im(A)^\perp$.

answer. $x \in Ker(A^T) \Leftrightarrow \forall x, < Ax, y >= 0 \Leftrightarrow y \in ImA^\perp$.

433. State Fredholm alternative verbally.

<u>answer</u>. Either the linear equation has a solution for a given data or else the homogeneous adjoint problem has a nontrivial solution and this solution is orthogonal to the given data.

2019-05-26

434. An n-layered fluid would have a barotropic mode and n-1 baroclinic modes. A continuously stratified fluid essentially has an infinite number of layers, and so has an infinite number of baroclinic modes.

435. If $b \in Ker(A^T) \setminus \{0\}$, can you solve $Ax = b$?

<u>answer</u>. No! To see, $0 = (x, A^T b) = (Ax, b) = (b, b) = |b|^2$. This is also a consequence of $Ker(A^T) = Im(A)^\perp$.

436. Compute directly the flux of $\hat{\mathbf{r}}/r^2$ over a sphere of radius R.

<u>answer</u>.
$$\oint \mathbf{v} \cdot d\mathbf{a} = \int \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot \left(R^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}\right)$$
$$= \text{area of unit sphere} = 4\pi$$

Second way: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$ and use divergence theorem.

437. What is an implicit solution to a first order ODE?

<u>answer</u>. $G(x, y) = 0$ is an implicit solution to the ODE $\frac{dy}{dx} = f(x, y)$ if there is a function $y = g(x)$ which satisfies both the ODE and functional relation.

<u>example</u>. $G(x, y) = x^2 + y^2 - c = 0$, $c > 0$ is an implicit solution to $\frac{dy}{dx} = -\frac{x}{y}$. Why? Because suppose $G_y(x_0, y_0) = 2y_0 \neq 0$ then by IFT $y = g(x)$ for $x \in I$ for some I containing x_0 and $\frac{dg}{dx} = -\frac{G_x(x,g(x))}{G_y(x,g(x))} = -\frac{x}{g(x)}$. That is there exists a function satisfying both equations.

438. What is the simplest configuration that yields a baroclinic QG structure?

<u>answer</u>. Two-layer system.

439. (Bathymetry) two-dimensional bump-like topography

$$h_B(x, y) = h_0 \cos(k_t x) \cos(l_t y)$$
$$h_B(x, y) = h_0 \sin(k_t x) \sin(l_t y)$$

where $h_B(x, y)$ denotes the topographic height above the resting depth of the fluid, h_0 the maximum/minimum amplitude and k_t and l_t the topographic wavenumbers in the $x-$ and y -direction.

440. (Rossby waves) The ocean responds to changes in atmospheric forcing via the planetary (Rossby) waves, which propagate westward across the basin [Anderson and Gill, 1975]. These have scales of hundreds to thousands of kilometers and are clearly observed in satellite measurements of sea surface height (SSH) [Chelton and Schlax, 1996]. LaCasce and Pedlosky [2004] suggested these waves are unstable, breaking into smaller, deeper eddies. This would have an enormous impact on oceanic adjustment, making it much more turbulent.

441. The bathymetry was excluded from the ocean theory in the past because the general thinking was that (*) is weak, so the (**) is probably also weak.

answer. (*) deep motion, (**) topographic influence.

Extra: Much ocean theory in the past, though, has ignored bathymetry, treating the bottom as a smooth, flat surface. For example, the models of Stommel [1948], Munk [1950], Stommel and Arons [1960], Anderson and Gill [1975] and Fofonoff [1954] all assumed a flat bottom. However, recent work (e.g. de La Lama et al. [2016] and LaCasce [2017]) suggests topography influences the ocean response throughout the water column further and significantly the vertical structure of the flow.

2019-05-25

442. (Sylvester's Theorem). Every square symmetric invertible matrix is congruent to a diagonal matrix with entries ± 1. The number of positive (resp. negative) entries is equal to the number of positive (resp. negative) eigenvalues. Prove it!

answer. By spectral theorem, $A = QDQ^{-1}$. $D = \mathrm{diag}(\lambda_i)$. As A is invertible, $\lambda_i \neq 0$. Let $U = \mathrm{diag}(\sqrt{|\lambda_i|})$. Then $A = Q^T U^T D' U Q$ where $D' = \mathrm{diag}(\mathrm{sign}(\lambda_i))$.

443. Let A be a bounded region in $\mathbb{R}^2$. $\oint_{\partial A} Pdx + Qdy = ?$

answer. $\iint_A (Q_x - P_y) dxdy$. Since for $\omega = Pdx + Qdy$, $d\omega = (Q_x - P_y)dxdy$ and the result follows from Stokes Theorem.

444. $\iint_A \nabla \times \mathbf{F} \cdot \mathbf{k} dA = \int_{\partial A} ?$ where $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and A is a bounded region in $\mathbb{R}^2$.

answer. Circulation. $\oint_{\partial A} \mathbf{F} \cdot d\mathbf{r}$

445. $\oint_{\partial A} \mathbf{F} \cdot d\mathbf{r} = \iint_A ?$ where $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and A is a bounded region in $\mathbb{R}^2$.

answer. $\iint_A \nabla \times \mathbf{F} \cdot \mathbf{k} dA$

446. If C is the boundary curve of A in the plane $\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint ?$

answer. $\iint_A \mathrm{div}\, \mathbf{F} dA$

447. (Manifold theory). What is a derivation?

answer. $D : C^\infty(M) \to C^\infty(M)$ is linear and satisfies the Leibniz rule: $D(fg) = gDf + fDg$ where f, g are smooth real valued functions.

448. Does $-\Delta u = f$ in U, $\nabla u \cdot \mathbf{n} = g$ in ∂U have a unique solution? Prove.

answer. The solution is unique up to a constant if and only if the compatibility condition

$$\int_U f dV = \int_U -\nabla \cdot \nabla u dV = -\int_{\partial U} g dS$$

is satisfied. Take $w = u_1 - u_2$. Then $-\Delta w = 0$ and $0 = \int_U |\nabla w|^2 \, dV$. Hence w is constant.

449. Relate the Eulerian conservation of mass $0 = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u}$ and Lagrangian conservation of mass $\rho J = \rho_0$ or $\rho(\phi(a,t),t)J(\phi(a,t),t) = \rho_0(a)$.

answer.

$$0 = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \frac{1}{J}\frac{DJ}{Dt} = \frac{1}{J}\frac{D}{Dt}(\rho J) \implies 0 = \frac{D}{Dt}(\rho J)$$

Recall that the solution of $\frac{Df}{Dt} = 0$ is $f(\phi(a,t),t) = f(\phi(a,0),0) = f_0(a)$. Thus

$$\rho(\phi(a,t),t)J(\phi(a,t),t) = \rho_0(a)J(a,0) = \rho_0(a)$$

which is the Lagrangian conservation of mass.

450. What is material derivative in Lagrangian coordinates?

answer. Time derivative with fixed a.

$$\frac{\partial F(a,s)}{\partial t} = \frac{d}{ds}\left(f(\phi(a,s),s)\right) = \left.\frac{Df(x,s)}{Dt}\right|_{x=\phi(a,s)}$$

where $f(\phi(a,t),t) = F(a,t)$.

To see,

$$\begin{aligned}
\frac{\partial F(a,s)}{\partial t} &= \frac{d}{ds}\left(f(\phi(a,s),s)\right) = f_t(\phi(a,s),s) + (\mathbf{U}(a,s)\cdot\nabla)f(\phi(a,t),t) \\
&= f_t(\phi(a,s),s) + (\mathbf{u}(\phi(a,s),s)\cdot\nabla)f(\phi(a,s),s) \\
&= \left.\frac{Df(x,s)}{Dt}\right|_{x=\phi(a,s)}
\end{aligned}$$

Example. $F(a,t) = at^2$, $\phi(a,t) = at$, $\phi^{-1}(x,t) = \frac{x}{t}$, $U(a,t) = a$, $u(x,t) = U(\phi^{-1}(x,t),t) = \frac{x}{t}$. Then

$$f(x,t) = F(\phi^{-1}(x,t),t) = F\left(\frac{x}{t},t\right) = xt$$

$$\frac{\partial F(a,s)}{\partial t} = 2as$$

$$\frac{Df(x,s)}{Dt} \mid_{x=\phi(a,s)} = x + \frac{x}{s} s \mid_{x=\phi(a,s)} = 2x \mid_{x=\phi(a,s)} = 2as$$

$$\frac{\partial F(a,s)}{\partial t} = \left.\frac{Df(x,s)}{Dt}\right|_{x=\phi(a,s)}$$

451. What kind of relation is the equation of state for sea water?

answer. $\rho = \rho(T, S, p)$

T = Temperature: ocean range: $-2°$C to $30°$C

S = Salinity = mass of salt (gm) dissolved in 1 kg seawater

http://mason.gmu.edu/~bklinger/seawater.pdf

452. Is sea water essentially compressible or incompressible?

answer. Incompressible.

2019-05-24

453. For barotropic incompressible Euler's equation under conservative forces, what can be said about the stability of irrotational flows in (A) 2D, (B) 3D case?

answer. In 2D case, $\frac{D\omega}{Dt} = 0$ vorticity is a conserved quantity, so if initial vorticity is small, it will always stay small. This is not the case as in 3D case $\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u}$. An initially irrotational flow always will stay irrotational but if slightly preturbed, vorticity may grow.

454. For 2D viscid incompressible barotropic flows under conservative forces, derive the vorticity equation.

answer. $\frac{D\omega}{Dt} = \nu \Delta \omega$ since in this case the equations of motion $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} + \nabla P$ and $\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = (\mathbf{u} \cdot \nabla)\omega$.

455. Expand $\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u})$.

answer. $\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$

proof.

(1) $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla \left(\frac{v^2}{2}\right) - \mathbf{u} \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{u}$

(2) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B}$

(3) $\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$

456. Which terms remain in the expansion of $\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u})$ when (A) $\nabla \cdot \mathbf{u} = 0$, (B) in 2D?

<u>`answer`</u>. The expansion is (see ENCUNU)

$$\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$$

(A)

$$\nabla \cdot \mathbf{u} = 0 \implies \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$$

(B)

$$\text{In 2D} \qquad \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla)\boldsymbol{\omega}$$

since $\boldsymbol{\omega}(x, y) = (0, 0, \omega(x, y))$ and $\mathbf{u}(x, y) = (u_1(x, y), u_2(x, y), 0)$.

457. Write the second order central scheme for second derivative $f''(x)$.

<u>`answer`</u>. $f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$. Why h^2?

458. Give 3 examples of barotropic stratification.

<u>`answer`</u>. (1) Ideal gas with constant temperature, (2) a fluid where pressure and density are functions of elevation only, (3) A homogeneous fluid.

459. Let

$$\Omega^1 = \{ f d\theta \mid f : S^1 \to \mathbb{R}, \text{ i.e. } f(0) = f(2\pi) \}$$

$$T : \Omega^1 \to \mathbb{R}, \qquad T : f d\theta \to \int_{S^1} f d\theta = \int_0^{2\pi} f(\theta) d\theta$$

T is a linear map between vector spaces Ω^1 and $\mathbb{R}$.

(1) What is $\operatorname{Ker} T$? Why?

(2) What is $\operatorname{Dom} T$? Why?

<u>`answer`</u>. (1) $\operatorname{Ker} T = \Omega^1_{\text{exact}}$ because if $\int_0^{2\pi} f(\theta) d\theta = 0$ then the scalar function $g(\theta) = \int_0^{\theta} f(\theta') d\theta'$ is a 0-form on S^1 since $g(0) = g(2\pi)$ and $dg = f d\theta$.

(2) $\operatorname{Dom} T = \Omega^1 = \Omega^1_{\text{closed}}$ since every 1-form on S^1 is closed.

460. Show that $H^1_{dR}(S^1) \cong \mathbb{R}$.

<u>`answer`</u>. Let

$$\Omega^1 = \{ f d\theta \mid f : S^1 \to \mathbb{R}, \text{ i.e. } f(0) = f(2\pi) \}$$

$$T : \Omega^1 \to \mathbb{R}, \qquad T : f d\theta \to \int_{S^1} f d\theta = \int_0^{2\pi} f(\theta) d\theta$$

T is a linear map between vector spaces Ω^1 and $\mathbb{R}$.

- $\operatorname{Ker} T = \Omega^1_{\text{exact}}$ because if $\int_0^{2\pi} f(\theta)d\theta = 0$ then the scalar function $g(\theta) = \int_0^{\theta} f(\theta')d\theta'$ is a 0-form on S^1 since $g(0) = g(2\pi)$ and $dg = f d\theta$.
- $\operatorname{Dom} T = \Omega^1 = \Omega^1_{\text{closed}}$ since every 1-form on S^1 is closed.
- $\operatorname{Im} T = \mathbb{R}$. This is obvious since $T(\frac{c}{2\pi}d\theta) = c$, for any $c \in \mathbb{R}$.

Hence by the definition of de-Rham cohomology and the first isomorphism theorem,

$$H^1_{dR}(S^1) = \Omega^1_{\text{closed}}/\Omega^1_{\text{exact}} = \operatorname{Dom}(T)/\operatorname{Ker}(T) \cong \operatorname{Im}(T) \cong \mathbb{R}.$$

461. What is a direct method to prove that de-Rham cohomology $H^k(M)$is a given vector space V?

 answer. Construct map $T : \Omega^k(M) \to W$ whose domain is closed k-forms, kernel is exact k-forms, and image isomorphic to V.

462. What is $H^1_{dR}(M)$ if M is simply-connected?

 answer. $\{0\}$ since any closed 1-form on a simply-connected space is exact.

 Also, integrals of closed 1-forms over homotopic paths are equal. And in a simply-connected space, any loop is homotopic to a trivial loop. Thus closed 1-forms are path-independent and thus exact.

463. If A^{-1} exists, what is $\frac{d}{dt}\det(A)$?

 answer. $\frac{d}{dt}\det(A) = \det(A)\operatorname{tr}\left(\frac{dA}{dt}A^{-1}\right) = \det(A)\operatorname{tr}\left(A^{-1}\frac{dA}{dt}\right)$. Note $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

464. In a barotropic flow, density is a function of pressure and also pressure is a function of density!

 answer. True.

465. Is a homogeneous density fluid always barotropic or baroclinic?

 answer. Barotropic.

2019-05-23

466. Define functional independence of k functions in a domain.

 answer. If their gradients are linearly independent at each point. Equivalently the matrix of gradients has full rank at each point.

467. Give a condition so that three functions on $\mathbb{R}^3$ are functionally independent.

 answer. $\nabla f_1 \cdot (\nabla f_2 \times \nabla f_3) \neq 0$ otherwise say $\nabla f_3 = c_1 \nabla f_1 + c_2 \nabla f_2$.

468. When do level surfaces of two functions on $f, g : \mathbb{R}^n \to \mathbb{R}$ coincide?

<u>answer</u>. When $\nabla f = \lambda \nabla g$ and $\nabla f \neq 0$, $\nabla g \neq 0$, that is f and g are functionally independent. In $\mathbb{R}^3$ this is identical to $\nabla f \times \nabla g = 0$.

469. When do level surfaces of two functions on $f, g : \mathbb{R}^3 \to \mathbb{R}$ coincide?

<u>answer</u>. When $\nabla f \times \nabla g = 0$ and $\nabla f \neq 0$, $\nabla g \neq 0$.

470. Give a condition so that two functions on $\mathbb{R}^3$ are functionally independent. What does the condition tell?

<u>answer</u>. $\nabla f_1 \times \nabla f_2 \neq 0$ since two functions are functionally dependent if $\nabla f_1 = k \nabla f_2$.

This means local level surfaces of the functions do not coincide.

471. In 3D, write $\nabla F_1 \times \nabla F_2$ as Jacobians.

<u>answer</u>. $= \left(\frac{\partial(F_1,F_2)}{\partial(y,z)}, \frac{\partial(F_1,F_2)}{\partial(z,x)}, \frac{\partial(F_1,F_2)}{\partial(x,y)} \right)$.

472. Show that barotropic (consider both $p = P(\rho)$ and $\rho = R(p)$ cases) flows are isentropic that is there exists w such that $\nabla w = \frac{1}{\rho(p)} \nabla p$.

<u>answer</u>. 1st case: Assume $p = p(\rho)$ and define $w = \int^{\rho} \frac{p'(\lambda)}{\lambda} d\lambda$. Then $w_x = \frac{p'(\rho)}{\rho} \rho_x$ and $\nabla w = \frac{p'(\rho)}{\rho} \nabla \rho = \frac{1}{\rho} \nabla p(\rho)$.

2nd case: $\rho = \rho(p)$ and define $w = \int^{p} \frac{1}{\rho(\lambda)} d\lambda$. Then $\nabla w = \frac{1}{\rho(p)} \nabla p$.

473. If $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x) = h(g(x))$ then

$$\nabla f \times \nabla g = 0$$

since $\nabla f = h'(g(x)) \nabla g$. Is the converse true?

<u>answer</u>. If $\nabla f \neq 0$ and $\nabla g \neq 0$, then LOCALLY this is true. For the proof see Advanced Calculus 3rd Edition by Taylor pg 264-265. The proof is nice. ToDO.

474. True or False? Constant density fluids are isentropic.

<u>answer</u>. True. A fluid is isentropic when there is enthalpy function w with $\frac{1}{\rho} \nabla p = \nabla w$.

475. Write the rate of change of circulation of velocity on a material line for 3D NSE. When is it zero (Kelvin's Theorem)?

<u>answer</u>. For

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} + \mathbf{F}$$

$$\frac{d}{dt} \int_{C_t} \mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{x} = \int_{C_t} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{x}$$

This is zero for isentropic ($-\frac{1}{\rho}\nabla p = \nabla w$), inviscid ($\nu = 0$) flows under conservative forces ($\mathbf{F} = \nabla f$).

476. (Kelvin's Circulation Theorem) Prove that if

$$\frac{D\mathbf{u}}{Dt} = \nabla w + \nabla f$$

then

$$\frac{d}{dt}\oint_{C_t} \mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{x} = 0$$

(isentropic flow under consevative force)

answer. By transport theorem for curves

$$\begin{aligned}\frac{d}{dt}\Gamma(t) &= \int_{C_t} \frac{D}{Dt}\mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{x} + \int_{C_t} \frac{1}{2} d(|\mathbf{u}|^2) \\ &= \int_{C_t} \frac{D}{Dt}\mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{x} = \int_{C_t} \nabla(w + f) \cdot d\mathbf{x} = (w + f)\mid_{C_t} = 0.\end{aligned}$$

since $d\left|\mathbf{u}\right|^2 = \nabla\left|\mathbf{u}^2\right| \cdot d\mathbf{x}$.

477. State the transport theorem for curves, $\frac{d}{dt}\int_{C_t} \mathbf{g}(\mathbf{x}, t) \cdot d\mathbf{x} =?$

answer.

$$\frac{d}{dt}\int_{C_t} \mathbf{g}(\mathbf{x}, t) \cdot d\mathbf{x} = \int_{C_t} \frac{D}{Dt}\mathbf{g}(\mathbf{x}, t) \cdot d\mathbf{x} + \mathbf{g}(\mathbf{x}, t) \cdot d\mathbf{u}.$$

since

$$\frac{d}{dt}\int_{C_t} \mathbf{g}(\mathbf{x}, t) \cdot d\mathbf{x} = \frac{d}{dt}\sum_i \mathbf{g}_i \cdot \Delta\mathbf{x}_i = \sum_i (\frac{D\mathbf{g}_i}{Dt} \cdot \Delta\mathbf{x}_i + \mathbf{g}_i \cdot \Delta\mathbf{u}_i)$$

478. Write the equation of motion for an inviscid, isentropic fluid in the presence of conservational body forces.

answer.

$$\frac{D\mathbf{u}}{Dt} = -\nabla w + \nabla f$$

w is the enthalpy, $\nabla w = \frac{1}{\rho}\nabla p$.

479. What is an isentropic flow? What is the origin of the name?

answer. $\frac{1}{\rho}\nabla p = \nabla w$. The scalar w is called enthalpy.

This terminology comes from thermodynamics, which state that

$$dw = T ds + \frac{1}{\rho} dp$$

where s is the entropy. If s is constant, hence the name isentropic.

480. For $D \in \mathbb{R}^2$, find the form of Green's function $\Delta_x G(x - x') = \delta(x - x')$ in D and $G = 0$ on ∂D.

<u>answer</u>. Let $G(x, x') = \frac{1}{2\pi} \ln(|x - x'|) + h(x)$ where $\Delta h = 0$ on D and $h(x) = \frac{-1}{2\pi} \ln(x - x')$ on ∂D.

481. $X_{\{f,g\}}$ in terms of the commutator?

<u>answer</u>. $X_{\{f,g\}} = [X_f, X_g]$

2019-05-22

482. Spectral theorem (Hilbert-Schmidt Theorem) in infinite dimensional spaces? (1) Space, (2) Operator, (3) Basis, (4) Properties of eigenvalues.

<u>answer</u>. Suppose A is a linear, compact, self-adjoint operator on an infinite dimensional (real or complex) Hilbert space H. Then there is an orthonormal basis of $R(A)$ consisting of eigenvectors of A, that is

$$Au = \sum_{i=1}^{N} \lambda_i \langle \varphi_i, u \rangle \varphi_i \text{ for all } u \in H.$$

Each eigenvalue is real. If eigenvalues are ordered so that

$$|\lambda_{n+1}| \leq |\lambda_n|$$

then

$$\lim_{n \to \infty} \lambda_n = 0$$

483. What is $H^0_{\mathrm{dR}}(M)$ when M has k connected components?

<u>answer</u>. $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^k$.

484. Show that $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^k$ when M has k connected components.

<u>answer</u>. Let $M = \cup_{i=1}^k M_i$. Then the functions χ_{M_i} form a basis for the vector space of closed 0-forms. The only exact 0-form is the zero function.

485. What is $H^0_{\mathrm{dR}}(M)$ when M is connected?

<u>answer</u>. $H^0_{\mathrm{dR}}(M) = \mathbb{R}$ since closed 0-forms are constants, a 1-d vector space and exact 0-forms is trivial space.

486. Define the k-th de Rham cohomology space.

<u>answer</u>. The vector space $H^k_{\mathrm{dR}}(M) = \frac{\{\text{ closed differential k forms on M}\}}{\{\text{ exact differential k forms on M}\}}$. It is equivalence classes of closed forms which differ by an exact form.

487. For which A is $x \to Ax$ a linear vector field on S^{n-1}?

<u>answer</u>. When A is skew-symmetric.

488. Why, for skew-symmetric A, $x \to Ax$ a linear vector field on S^{n-1}?

<u>`answer`</u>. Since it that case $\langle x, Ax \rangle = 0$ which means Ax is tangent to S^{n-1}. Or alternatively, $e^{tA} \in SO(n)$ and $e^{tA}x_0 \in S^{n-1}$ if $x_0 \in S^{n-1}$.

489. $\langle x, Ax \rangle = 0$ if and only if A is ...

<u>`answer`</u>. skew-symmetric. proof not hard. `http://www-math.mit.edu/~larsh/teaching/vectorfields.pdf`.

490. Radial harmonic functions on $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$?

<u>`answer`</u>. In $\mathbb{R}^2$: $a \ln r + c$, In $\mathbb{R}^n$: $ar^{2-n} + b$, $n \geq 3$.

491. Is angle dimensional?

<u>`answer`</u>. No. Angle of an arc is arc length over radius which is a ratio of two length units.

492. Give an example to show that angle is non-dimensional.

<u>`answer`</u>. Length of arc of radius r and angle θ is $r\theta$. Since $r\theta$ has dimension length then θ must be non-dimensional.

493. State the spectral theorem for normal matrices and compare with Hermitian case.

<u>`answer`</u>. For normal matrices we have

$$A = UDU^*$$

where D is a diagonal matrix of eigenvalues of A and U is an unitary matrix of corresponding eigenvectors. The difference with the Hermitian case is D may be complex.

494. Show that if all the eigenvalues of a normal matrix A are real then A is Hermitian.

<u>`answer`</u>. By spectral theorem, we have $A = UDU^*$ where D is a diagonal matrix of eigenvalues of A and U is an unitary matrix of corresponding eigenvectors. Since D is real, $A^* = UD^*U^* = A$.

495. State the spectral theorem for symmetric and hermitian matrices.

<u>`answer`</u>. *Real Case.* Any symmetric real matrix A can be diagonalized by an orthogonal matrix, that is $D = Q^T AQ$ where D is a real diagonal matrix of eigenvalues.

Complex Case. Any Hermitian matrix A can be diagonalized by an unitary matrix, that is $D = Q^T AQ$ where D is a real diagonal matrix of eigenvalues.

496. What is the difference between orthogonal and unitary matrices?

answer. For orthogonal matrices $QQ^T = I$ and for unitary matrices $UU^* = I$ where $*$ conjugate transpose. For real matrices, a matrix is unitary if and only if it is orthogonal.

497. Do eigenvectors of a Hermitian matrix must be real as well? What is the property of the matrix of eigenvectors?

answer. No they only form a unitary matrix (meaning $v_i \cdot v_j^* = \delta_{ij}$) . Take $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$. Then

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix} (\lambda_1 = -1), \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix} (\lambda_2 = 6), \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix} (\lambda_3 = -2).$$

Checked by Mathematica as well.

498. 2019-05-21

499. Write the 5-point stencil finite-difference scheme for the 2D-Laplacian.

answer. This can be seen easily when considering a finite-difference approximation to the Laplacian:

$$\nabla^2 f(x,y) \approx \frac{f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h) - 4f(x,y)}{h^2} = \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} + \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2} = \frac{1}{h^2}\sum_{\mathbf{h}} f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})$$

This describes the averaging property of Laplacian. Why h^2?

500. What is the idea that every closed 1-form on a simply-connected space is exact?

answer. $\int_C \alpha$ depends only on the end points by Stokes Theorem since $\int_{\partial R} \alpha = \iint_R d\alpha = 0$ since α is closed. Thus the function $f(P) = \int_{P_0}^{P} \alpha$ is well-defined and is an anti-integral.

501. What can be said about the curl of a radial vector field?

answer. It is zero.

proof. Let $\mathbf{v} = f(r)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. By symmetry, it suffices to check $v_{3,y} - v_{2,z} = 0$. $v_3 = f(r)z$ and $v_{3,y} = f'(r)\frac{y}{r}z$ and $v_{2,z} = f'(r)\frac{z}{r}y$.

502. Expand $\vec{\nabla} \cdot (\vec{A} \times \vec{B})$

answer. $(\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B})$

503. Write the integration by parts $\iiint \mathbf{u} \cdot (\nabla \times \mathbf{v}) dV$.

answer. $\iiint \mathbf{u} \cdot (\nabla \times \mathbf{v}) dV = \iiint (\nabla \times \mathbf{u}) \cdot \mathbf{v} dV + \oint \mathbf{u} \cdot (\mathbf{n} \times \mathbf{v}) dS$.

Proof is not important. $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$.

By divergence theorem, $\oint (\mathbf{u} \times \mathbf{v}) \cdot d\mathbf{S} = \iiint (\nabla \times \mathbf{u}) \cdot \mathbf{v} dV - \iiint \mathbf{u} \cdot (\nabla \times \mathbf{v}) dV$.

Finally, by the triple product propert (just determinant), $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} = -\mathbf{u} \cdot (\mathbf{n} \times \mathbf{v})$.

504. Write the integration by parts $\iiint u(\nabla \times \mathbf{v}) dV$.

answer. $$\iiint u(\nabla \times \mathbf{v}) dV = \oint u(\mathbf{n} \times \mathbf{v}) dS - \iiint (\nabla u \times \mathbf{v}) dV.$$

ToDO: Can this be expressed by differential forms?

Reason (proof is not important, remember the form). In $\iiint \mathbf{u} \cdot (\nabla \times \mathbf{v}) dV = \iiint (\nabla \times \mathbf{u}) \cdot \mathbf{v} dV - \oint (\mathbf{u} \times \mathbf{v}) \cdot d\mathbf{S}$, take $\mathbf{u} = u\mathbf{e}$ where $\mathbf{e}$ is arbitrary and constant. Use (1) $(\nabla \times \mathbf{u}) \cdot \mathbf{v} = (\nabla u \times \mathbf{e}) \cdot \mathbf{v} = -\mathbf{e} \cdot (\nabla u \times \mathbf{v})$ (think triple product as determinant), (2) $\mathbf{n} \cdot (u\mathbf{e} \times \mathbf{v}) = -\mathbf{e}u \cdot (\mathbf{n} \times \mathbf{v})$

$$\mathbf{e} \cdot \iiint u(\nabla \times \mathbf{v}) dV = -\mathbf{e} \cdot \iiint (\nabla u \times \mathbf{v}) dV + \mathbf{e} \cdot \oint u(\mathbf{n} \times \mathbf{v}) dS$$

505. What is the physical meaning of $c_t + \nabla \cdot (c\mathbf{v}) = 0$?

answer. This means c is a conserved quantity transported by $\mathbf{u}$ and there is no diffusion and no source/sink. The equation can be derived from $0 = \frac{d}{dt} \int_{W_t} c dV = \int_{W_t} (c_t + \nabla \cdot (c\mathbf{v})) dV$ where W_t is a material region.

2019-05-20

506. Show that the Poisson equation has at least one solution on $\mathbb{R}^n$ if we look for solutions that vanish at infinity.

answer. We need to show that the particular solution $\int_{\mathbb{R}^n} G(x-x')f(x')dx'$ vanishes at infinity.

507. Show that Poisson equation $\Delta u = f$ has at most one solution on $\mathbb{R}^n$ if we look for solutions that vanish at infinity.

answer. If there are two solutions u_1, u_2 then $u = u_1 - u_2$ is harmonic and vanishes at infinity. Since there is no non-trivial harmonic function that vanishes at infinity, $u \equiv 0$.

508. What condition has to be put so that the inverse Laplace Δ^{-1} is well-defined on $\mathbb{R}^n$?

<u>answer</u>. We can put the condition that we are looking for a function which vanishes at infinity $\lim_{|x|\to\infty} u(x) = 0$. There are no harmonic functions which vanish at infinity. Still incomplete.

509. Is the inverse of the Laplace operator defined on $\mathbb{R}^n$ when no condition for the inverse is specified. Why?

<u>answer</u>. Because if h is a harmonic function on $\mathbb{R}^n$ such as $h(x) = x_i$ and $\Delta u = f$ then $\Delta(u + h) = f$ and Δ^{-1} is not well-defined.

510. Show uniqueness for $-\Delta u = f$ in U and $u = g$ on ∂U. (Extra: Prove in two different ways.)

<u>answer</u>. <u>1st way</u>. Maximum principle: The difference of two solutions satisfies the homogeneous equation and homogeneous BC which must be zero by the max principle. <u>2nd way</u>. Energy method: $w = u_1 - u_2$. $0 = \int_U \Delta w w dV = -\int_U |\nabla w|^2 dV$. Hence $w = c$ in U and by the boundary conditions $w = 0$.

511. Prove that $KerA^T = (ImA)^{\perp}$.

<u>answer</u>. $y \in KerA^T \iff (Ax, y) = 0, \quad \forall x \iff y \in (ImA)^{\perp}$.

512. How to define the differential of a map $f : M \to N$ using the derivation definition of tangent space?

<u>answer</u>. For $g : N \to \mathbb{R}$ define $df_p(X)(g) = X(g \circ f)$.

513. In 2D, give 3 classes of harmonic functions. Which of them satisfy the condition $\lim_{|x|\to\infty} h(x) = 0$?

<u>answer</u>. Polynomials: $a(x^2 - y^2) + bxy$, radially symmetric: $a \ln r + b$, exponentials: : $e^{kx} \sin(ky), \ldots$.

None of these functions satisfy $\lim_{|x|\to\infty} h(x) = 0$ except for $h(x) = 0$.

514. State the conservation of mass in Lagrangian form.

<u>answer</u>. $\rho_0(a) = \rho(\phi(a, t), t) \det\left(\frac{\partial\phi(a,t)}{\partial a}\right) = \rho(\phi(a, t), t) j(\phi(a, t), t)$.

<u>To see</u>, taking derivative w.r.t t

$$0 = \frac{D}{Dt}(\rho(\cdot) j(\cdot)) = j(\cdot)\frac{D\rho(\cdot)}{Dt} \implies \frac{D\rho(\cdot)}{Dt} = 0$$

515. Name the equation $\rho_0(a, t) = \rho(\phi(a, t)) \det\left(\frac{\partial\phi(a,t)}{\partial a}\right)$.

<u>answer</u>. Conservation of mass in Lagrangian form.

516. For an incompressible flow, the density is constant. True?

answer. No. Only material density is constant or divergence of flow is zero.

517. What is the definition of an incompressible flow? Extra: Give two other equivalent conditions.

answer. Three equivalent definitions: (1) $D\rho/Dt = 0$ (2) $\nabla \cdot \mathbf{u} = 0$, (3) $J = \det\left(\frac{\partial\phi(a,t)}{\partial a}\right) \equiv 1$.

2019-05-19

518. What is the Euler characteristic definition for a polyhedron?

answer. Let P be a polyhedron with V vertices, E edges, and F faces. Then we define the Euler characteristic to be $\chi_E(P) = V - E + F$.

519. "Distinct air mass regions exist. Fronts separate warmer from colder air." Is this barotropic or baroclinic?

answer. Baroclinic.

520. "Region of uniform temperature distribution; A lack of fronts." Is this barotropic or baroclinic?

answer. Barotropic.

521. Are tropics barotropic or baroclinic?

answer. Barotropic

522. There are clear density gradients in a X environment caused by the fronts. Is X barotropic or baroclinic?

answer. Baroclinic

523. A X atmosphere is out of balance. Is X baroclinic or barotropic?

answer. Baroclinic. Part of the word baroclinic is clinic. If the atmosphere is out of balance, it is baroclinic, just as if a person felt out of balance they would need to go to a clinic.

524. A mid-latitude cyclone is a X environment. Is X baroclinic or barotropic?

answer. Baroclinic.

525. Define Laplacian using scale factors.

answer.

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u^2}\right) + \frac{\partial}{\partial u^3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u^3}\right)\right]$$

526. Define curl using scale factors.

answer.

$$\nabla \times \mathbf{v}_R = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\boldsymbol{e}}_1 & h_2 \hat{\boldsymbol{e}}_2 & h_3 \hat{\boldsymbol{e}}_3 \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

527. Define div using scale factors.

answer.

$$\nabla \cdot V = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u^1} (V_1 h_2 h_3) + \frac{\partial}{\partial u^2} (h_1 V_2 h_3) + \frac{\partial}{\partial u^3} (h_1 h_2 V_3) \right]$$

528. Define grad using scale factors.

answer.

$$\nabla = \frac{\hat{\mathbf{e}}_1}{h_1} \frac{\partial}{\partial u^1} + \frac{\hat{\mathbf{e}}_2}{h_2} \frac{\partial}{\partial u^2} + \frac{\hat{\mathbf{e}}_3}{h_3} \frac{\partial}{\partial u^3}$$

2019-05-18

529. Estimate the u_r, u_{rr} and $u_{\theta\theta}$ coefficients in polar Laplacian by dimensional analysis.

answer. $\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$. The terms have the same dimension U/L^2 and angle is dimensionless.

530. Write the differential line element in spherical coordinates.

answer. $d\boldsymbol{l} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$.

531. Find the motion and its inverse when the (Eulerian) velocity is constant.

answer. $\phi(\mathbf{a}, t) = \mathbf{a} + t\mathbf{u}$, $\phi^{-1}(\mathbf{x}, t) = \mathbf{x} - t\mathbf{u}$.

2019-05-17

532. Every symmetric matrix is up to choice of an ..., a diagonal matrix.

answer. orthonormal basis

533. What is the inertial acceleration term in fluid equations and what is its units?

answer. $|\mathbf{v} \cdot \nabla \mathbf{v}| \sim U^2/L$.

534. Which equations can describe atmospheric and oceanographic flows and why?

answer. Shallow water equations.

535. Two geometric meanings of gradient?

answer. the direction of steepest ascent, normal of the level surfaces.

536. What does divergence describe?

answer. amount of stuff created at a point

537. What does Laplacian describe?

answer. the average rate of change at a point.

538. In image processing, a discrete Laplacian can be used as a crude ... filter. It is close to zero in regions where the image is varying smoothly, and has large values in regions where the image has sharp transitions from low to high intensity.

answer. edge-detection

539. Show that on a compact manifold without boundary, every harmonic function is constant.

answer. A harmonic function can not have any max or min in there interior of its domain. But a continuous function on a compact domain must have a max and min.

540. Show that if $\omega = \nabla \times \mathbf{u}$ then there is $\mathbf{v}$ with $\omega = \nabla \times \mathbf{v}$ with $\nabla \cdot \mathbf{v} = 0$.

answer. Idea is that addition of any gradient term to the vector potential does not change the curl.

$\omega = \nabla \times (\mathbf{u} + \nabla f)$ for any f. Then $\nabla \cdot (\mathbf{u} + \nabla f) = 0$ means $\Delta f = -\nabla \cdot \mathbf{u}$. Solve for f to obtain $\mathbf{v}$.

This is known as **gauge selection**.

541. Express $\int_{\mathbb{R}^n} f(|x|)dx$ as a single integral.

answer. $\omega_{n-1} \int_0^\infty f(r) r^{n-1} dr$ where ω_{n-1} is the surface area of the unit n-sphere. To see this

$$\int_{\mathbb{R}^n} f(|x|)dx = \int_0^\infty f(r) \int_{S_{n-1}(r)} dS dr = \int_0^\infty f(r) \int_{S_{n-1}(1)} r^{n-1} dS dr$$

In 2D this is polar coordinates: $\int_0^{2\pi} \int_0^R f(r) r dr d\theta$. In 3D this is spherical coordinates.

542. Is every conservative ODE also volume-preserving? If not, find a counter-example.

answer. $\dot{x} = x$, $\dot{y} = y$ has a first integral $F(x,y) = y/x$ since $\dot{F} = 0$. However the system is not volume-preserving as its divergence is non-zero.

543. Unitary matrix means, columns form ...? answer. an orthonormal basis.

2019-05-16

544. What is the divergence of a radial vector field? Take the divergence of $\frac{\hat{r}}{r^2}$. Hint: use spherical divergence.

answer. $\vec{\nabla} \cdot (f(r)\hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f(r)\right)$.
$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$.

545. State the maximum principle.

answer. Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is harmonic and attains either a global minimum or maximum in Ω, then u is constant.

546. If u is a harmonic function defined on whole $\mathbb{R}^n$ with $\lim_{|x|\to\infty} u(x) = c$ then $u(x) = c$ everywhere. In particular, there is no nontrivial harmonic function which vanishes at infinity.

answer. Suppose u is a harmonic function on $\mathbb{R}^n$. Take $B(0, r) \subset \mathbb{R}^n$ a closed ball. Then the maximum and minimum values of u occur at the boundary. As $r \to \infty$, we see that u must be zero everywhere.

This is a special case of Liouville's Theorem which states that bounded harmonic functions defined on whole of $\mathbb{R}^n$ must be constant.

547. Liouville's Theorem says that harmonic functions defined on whole $\mathbb{R}^n$ must be bounded. Can one find a harmonic function defined on all $\mathbb{R}^n$ except a compact region?

answer. Yes! $\frac{1}{|\mathbf{x}-\mathbf{x}_0|}$ is a bounded harmonic function on $\mathbb{R}^n \setminus B_\epsilon(\mathbf{x}_0)$ for every $\epsilon > 0$.

2019-05-15

548. Are focuses possible in a gradient system?

answer. No. Eigenvalues of the linearized operator are real.

549. Classify the non-degenerate equilibria of a gradient system.

answer. $\dot{x} = -\nabla f(x)$ has a saddle where f has a non-degenerate saddle, stable node where f has a non-degenerate minimum and an unstable node where f has non-degenerate maximum.

To see. The linearization of a gradient system at an equilibrium $x = x_0$ is $\dot{x} = -\nabla f(x)$ is $\dot{y} = -D^2 f(x_0)y$. The eigenvalues of $-D^2 f(x_0)$ are real and nonzero since it is a symmetric, non-degenerate matrix. Thus either all eigenvalues are positive/negative and the equilibrium is a source/sink or some are positive and some are negative and the equilibrium is a saddle.

550. Research the degenerate equilibria of a gradient system. First the isolated case, second non isolated case. ToDO

551. Define a non-degenerate equilibrium of a function $f : \mathbb{R}^n \to \mathbb{R}$.

answer. $\nabla f(x_0) = 0$ and $\det D^2 f(x_0) \neq 0$.

552. What can be said about degenerate equilibria of a gradient system?

553. Define vector field on a manifold.

answer. $X : M \to TM$ such that $X(p) \in TM_p$. Note that $\pi \circ X = id_M$ where $\pi : TM \to M$ is the projection.

An alternative method is to define a vector field as a derivation on smooth real valued functions on M.

554. Give an example of a 2-form which is closed but not exact.

answer. $\star d\left(\frac{-1}{r}\right) = \star\left(\frac{xdx + ydy + zdz}{r^3}\right) = \frac{xdydz + ydzdx + zdxdy}{(x^2+y^2+z^2)^{3/2}}$ is a 2-form on $\mathbb{R}^3 - \{(0,0,0)\}$ which is closed (its divergence is zero) but not exact (its not the curl of another field).

The reason for the existence of such a 2-form is that this space is not contractible (although it is simply-connected) and thus the Poincaré Lemma does not hold (globally).

Note that $\mathbb{R}^3 \setminus \{0\}$ is homotopy equivalent to S^2 which has trivial first de-Rham cohomology but non-trivial (isomorphic to $\mathbb{R}$) second de-Rham cohomology.

555. In 3D, find $\star dx$, $\star dy$, $\star dz$, $\star(dx \wedge dy)$.

answer. $\star dx = dy \wedge dz$, $\star dy = dz \wedge dx$, $\star dz = dx \wedge dy$, $\star(dx \wedge dy) = dz$.

2019-05-14

556. What is a harmonic vector field?

answer. A vector field which is both irrotational and incompressible.

557. Show that the vector Laplacian of a harmonic vector field is zero.

answer. Since $\nabla^2 \mathbf{s} \equiv \nabla(\nabla \cdot \mathbf{q}) - \nabla \times (\nabla \times \mathbf{p}) = \mathbf{0}$.

558. When does a harmonic field have a potential? What is the property of the potential?

answer. When the domain is simply-connected, since $\mathbf{v}$ is irrotational, $\mathbf{v} = \nabla\phi$. Since $\mathbf{v}$ is divergence-free $\nabla^2\phi = 0$. The potential is harmonic.

559. Show that each component of a harmonic vector field is a harmonic function on a simply-connected domain.

answer. Since $\nabla \times \mathbf{v} = \mathbf{0}$, it follows that $\mathbf{v} = \nabla \phi$. Since $0 = \nabla \cdot \mathbf{v} = \nabla^2 \phi$. Thus ϕ and therefore $\frac{\partial \phi}{\partial x_i}$ are all harmonic.

560. Show that not every harmonic function is a component of harmonic vector field (although locally this is true).

answer.

2019-05-13

561. On $\mathbb{R}^2$, write the Laplacian in terms of Hodge star $\star$ and d. (Hint start with df).

answer. $\Delta = \star d \star d$. To see: Let $\mathrm{Vol} = dx \wedge dy$. Then $\star dx = dy$, $\star dy = -dx$. $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$. $\star df = \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx$. $d \star df = \frac{\partial^2 f}{\partial x^2} dx \wedge dy + \frac{\partial^2 f}{\partial y^2} dx \wedge dy$. $\star d \star df = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

2019-05-11

562. (Poincaré-Hopf). Let M be a compact oriented surface and $v : M \longrightarrow TM$ a smooth vector field with isolated zeros. What is the relation between the Euler characteristic of M and the indices of the zeros?

answer. The sum of the indices at the zeros equals the Euler characteristic of M:

$$\sum_{x \text{ zero of } v} ind_{x,v} = \chi(M)$$

563. (Hairy Ball Theorem) Show that any vector field on a 2-sphere must have singularities.

answer. Since the Euler characteristic of the sphere is 2 (non-zero), the sum of indices of singular points of any vector field must be 2. In particular, there must exist singular points.

564. Define the index of a critical point in 2D formally.

answer. Let $v = (v_1, v_2)$ be a vector field on a surface S, x an isolated singularity of v, θ be the angular coordinate of the vector field, that is $\theta = \arctan\left(\frac{v_2}{v_1}\right)$. The Poincaré index of v in x is

$$ind_{x,v} = \frac{1}{2\pi} \int_\gamma d\theta = \frac{1}{2\pi} \int_\gamma \frac{v_1 dv_2 - v_2 dv_1}{v_1^2 + v_2^2}$$

565. In n-dimensions, which closed k-forms and which exact k-forms are trivial?

answer. Closed 0-forms (functions which are constant on each connected component) and closed n-forms (every n-form) are trivial. Closed k-forms, $1 \leq k \leq n-1$ are interesting.

Only exact 0-forms are trivial which consist of 0 by convention.

566. $SO(n)$ is the connected-component of ... containing ....

answer. $O(n)$, identity.

567. When is the system $\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$ a gradient system?

answer. When the form is closed ($f_y = g_x$) and the domain is simply-connected.

568. On a gradient system $\dot{x} = -\nabla V(x)$, what can be said about $V(x(t))$?

answer. Non-increasing with strictly decreasing except at fixed points. $\frac{d}{dt}V(x(t)) = -\|\dot{x}\|^2 \leq 0$. V is non-decreasing along orbits and the system points in the direction of local minima (if any) of V.

2019-05-10

569. Why is $X_{\{f,g\}} = [X_f, X_g]$ where $[\cdot,\cdot]$ is the commutator and $\{\cdot,\cdot\}$ is the Poisson bracket?

answer. Follows from the Jacobi identity. $X_{\{f,g\}}(h) = \{\{f,g\},h\}$ and $[X_f, X_g](h) = \{f,\{g,h\}\} - \{g,\{f,h\}\} = -\{\{g,h\},f\} - \{\{h,f\},g\}$.

570. If H is the Hamiltonian and I is an integral of motion then what is zero?

answer. $X_I(H) = \{I, H\} = 0$

571. If H is the Hamiltonian and I is an integral of motion then what is $X_I(H)$?

answer. $X_I(H) = \{I, H\} = 0$.

572. What is the vector field X_f generated by the function f by means of Poisson bracket?

answer. $X_f(g) = \{f, g\}$.

573. For $\dot{x} = Ax$ where $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$, write the solution explicitly using the exponential matrix.

answer. Since $e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (see another question), the solution is

$$x = e^{tA}\mathbf{x}_0 = \begin{pmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{pmatrix}\mathbf{x}_0$$

574. Find a function u such that $\Delta u = f$ in $\mathbb{R}^3$. Is the solution unique?

<u>answer</u>. $u(x) = -\dfrac{1}{4\pi}\displaystyle\int \dfrac{f\left(x'\right)}{\left|x - x'\right|} d^3x'$

This function is not unique. For any harmonic function h, $\Delta(h + u) = f$.

575. What is $\dfrac{\partial r}{\partial x_i}$?

<u>answer</u>. $\dfrac{x_i}{r}$.

576. What can be said about the Hessian of a harmonic function?

<u>answer</u>. Symmetric traceless matrix.

577. Find all harmonic quadratic polynomials on $\mathbb{R}^3$.

<u>answer</u>. $ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j$ of the form $a + b + c = 0$ and d, e, f, g, h, i, j arbitrary.

578. With $\mathbf{u}$ as the velocity, write the differential form of (A) $\frac{d}{dt}\int_{W_t} f dV = 0$ and (B) $\frac{d}{dt}\int_W f dV = -\int_{\partial W} \mathbf{F}\cdot\mathbf{n} + \int_W Q dV$. (C) When do these two give the same result?

<u>answer</u>. (A) $f_t + \nabla\cdot(f\mathbf{u}) = 0$ and (B) $f_t + \nabla\cdot\mathbf{F} = Q$. (C) When $Q = 0$ and $\mathbf{F} = \mathbf{u}f$.

579. What is $\dot{J} = \frac{d}{dt}\det(\nabla_{\mathbf{a}}\mathbf{x})$? Prove.

<u>answer</u>. $J(\nabla\cdot\mathbf{u})$. <u>Reason</u>: $\frac{d}{dt}\det(\mathbf{F}) = \det(\mathbf{F})\,\mathrm{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1})$, $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ and $\mathbf{L} = \nabla\cdot\mathbf{u}$.

<u>2019-05-09</u>

580. What is Green's function to Laplace operator on $\mathbb{R}^2$ and $\mathbb{R}^3$?

<u>answer</u>. 2d: $\frac{1}{2\pi}\ln(x - x')$,
3d: $\frac{-1}{4\pi|x-x'|}$.

581. What is the relation between ρ, ρ_0, J, a and $x = \phi(a,t)$?

<u>answer</u>. $\rho_0(a) = \rho(x,t)J(x,t)$. This is the alternative form of the conservation of mass.

582. What is the Lagrangian differential form of the conservation of mass.

<u>answer</u>. $\frac{D(\rho J)}{Dt} = 0$ whose solution is $\rho(x,t)j(x,t) = \rho_0(a)J(a) = \rho_0(a)$.
$\rho_0 dV = \rho dv$ or $\rho_0 = \rho j$.

583. Obtain the 3D volume change $dv = JdV$ by using the fact that the volume is given by triple product.

<u>answer</u>. The edges of a infinitesimal parallelpiped is $d\mathbf{X}_i = dX_i\mathbf{E}_i$. The Lagrangian volume element is

$$dV = d\mathbf{X}_1 \cdot (d\mathbf{X}_2 \times d\mathbf{X}_3) = dX_1dX_2dX_3\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3) = dX_1dX_2dX_3.$$

Upon deformation, these edges go to

$$d\mathbf{x}_i = \boldsymbol{F} \cdot d\mathbf{X}_i = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \cdot d\mathbf{X}_i = dX_i\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \cdot \mathbf{E}_i = dX_i\frac{\partial \mathbf{x}}{\partial X_i}$$

The deformed volume is

$$dv = d\mathbf{x}_1 \cdot (d\mathbf{x}_2 \times d\mathbf{x}_3) = dX_1dX_2dX_3\frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3}\right)$$

Since $J = \det(\boldsymbol{F}) = \det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right) = \frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3}\right)$, the result follows.

By the conservation of mass $\rho_0 dV = \rho dv$ hence we get the alternative form the conservation of mass $\rho_0 = \rho J$.

584. Liouville's Theorem states that the density of particles in phase space is ... .

<u>answer</u>. constant in time

585. For a smooth function on $u : U \subset \mathbb{R}^n \to \mathbb{R}$ satisfying some BC what is $\int_U \Delta u(x')G(x-x')dx'$ where G is the Green's function to the Laplace operator on U with BC? Why?

<u>answer</u>. $u(x)$. To see, integrate by parts twice, the boundary terms are gone since u and G satisfy the same BC. Also $\Delta G(x - x') = \delta(x - x')$.

<u>2019-05-08</u>

586. What is the meaning of the transport equation $c_t + \mathbf{v} \cdot \nabla c = 0$ and more generally $c_t + \nabla \cdot (c\mathbf{v}) = 0$?

<u>answer</u>. The first one is the evolution of a conserved quantity c transported by a divergence free vector field $\mathbf{v}$. The second equation is the general case, $\mathbf{v}$ is not necessarily divergence-free.

587. For the transport equation $c_t + \mathbf{v} \cdot \nabla c = 0$, $c(0, x) = c_0(x)$, what is the solution when $\mathbf{v}$ is constant?

<u>answer</u>. $c_0(x - t\mathbf{v})$.

588. For the transport equation $c_t + \mathbf{v} \cdot \nabla c = 0$, $c(0, x) = c_0(x)$, FIND the solution when $\mathbf{v}$ is constant?

<u>answer</u>. Since $(1, \mathbf{v}) \cdot \left(\frac{\partial}{\partial t}, \nabla\right) c = 0$, c is constant in the direction $(t, x) = (1, \mathbf{v})$. Set $z(s) = c(t + s, x + s\mathbf{v})$ then $\frac{dz}{ds} = 0$. $c(t + s, x + s\mathbf{v}) = z(s) = z(0) = c(t, x)$. Hence $c(s, x + s\mathbf{v}) = c_0(x)$. Write $x + s\mathbf{v} = x'$ and $s = t$ to get $c(t, x') = c_0(x' - t\mathbf{v})$.

2019-05-07

589. What is a symplectic manifold? What is this useful for?

<u>answer</u>. It is a smooth manifold equipped with a closed nondegenerate differential 2-form, called the symplectic form.

Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds. For example, in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field, the set of all possible configurations of a system is modeled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

2019-05-06

590. What is the PDE satisfied by the time dependent first integral Φ of the ODE $\dot{x} = \mathbf{u}(x, t)$?

<u>answer</u>. The transport equation

$$\Phi_t + \mathbf{u} \cdot \nabla\Phi = 0$$

or

$$\frac{D\Phi}{Dt} = 0$$

since

$$0 = \frac{d}{dt}\Phi(x(t), t) = \frac{D\Phi}{Dt}$$

591. What is the PDE satisfied by the first integral of the ODE $\dot{x} = \mathbf{f}(x)$? What is the physical meaning?

<u>answer</u>.

$$\nabla u \cdot \mathbf{f} = 0$$

or

$$f_1(x)u_{x_1} + f_2(x)u_{x_2} + \cdots + f_n(x)u_{x_n} = 0$$

where $\mathbf{f} = (f_1, \cdots, f_n)$.

The physical meaning is that, u is constant along $\mathbf{f}$ or u does not change in the direction of $\mathbf{f}$ or $\mathbf{f}$ is tangent to level sets of u.

2019-05-05

592. $\dot{x} = -x$, $\dot{y} = y + x^2$. What is the unstable space and unstable manifold?

answer. The y axis is both the unstable space and the unstable manifold.

593. $\dot{x} = -x$, $\dot{y} = y + x^2$. Find the stable manifold by series expansion.

answer. Let $y = h(x) = a_2 x^2 + \cdots$, using $h(x) + x^2 = \dot{y} = h'(x)\dot{x}$, that is

$$(a_2 + 1)x^2 + a_3 x^3 + \cdots = -2a_2 x^2 - 3a^3 x^3 - \cdots$$

and we find exactly that $y = h(x) = -\frac{x^2}{3}$. This invariant set is the stable manifold of the origin tangent at origin to stable space E_s which is the x-axis.

594. $\dot{x} = -x$, $\dot{y} = y + x^2$. What is the ODE satisfied by the stable manifold function?

answer. $\dot{y} = h'(x)\dot{x}$ or $h(x) + x^2 = -h'(x)x$.

595. What is the global attractor of $\dot{x} = x - x^3$? What does it consist of? Give an absorbing set.

answer. The set $A = [-1, 1]$ is the attractor. The attractor consists of 3 fixed points, two are stable one is unstable and connecting orbits. Note that the system is multi-stable. Any set $[-1-\delta, 1+\delta]$ is an absorbing set.

596. What is the inclusion relation between $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$?

answer. $GL(n,\mathbb{R}) \supset SL(n,\mathbb{R}) \supset SO(n) \subset O(n)$. By definition $SO(n) = O(n) \cap SL(n,\mathbb{R})$.

597. If $A \in SL(n,\mathbb{R})$, what can be said about stability of the zero solution of the system $\dot{x} = Ax$? Is the system volume preserving?

answer. Nothing. The eigenvalues satisfy $\lambda_1 \cdots \lambda_n = 1$. The origin can be stable/unstable/saddle. The system is volume preserving if $0 = \operatorname{div}(Ax) = \operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

598. For $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$ which groups are connected and if disconnected, how many connected-components are there?

answer. $GL(n,\mathbb{R})$ has two connected components and $SL(n,\mathbb{R})$ is the connected component containing the identity (see Marsden, Ratiu pg. 285). $O(n)$ has also two connected-components and $SO(n)$ is the component containing the identity.

599. Is the map $\exp : \mathfrak{o}(n) \to O(n)$ surjective?

answer. No. There are matrices in $O(n)$ with determinant -1. But the exponential of a skew-symmetric matrix $\det e^A = e^{\operatorname{tr}(A)} = 1$.

600. What is the relation between $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$? Why?

answer. They are same. Basically $SO(n)$ is the connected component of $O(n)$ containing the identity. So the tangent space at identity are equal in both cases.

601. Is $\exp : \mathfrak{sl}(n) \to SL(n, \mathbb{R})$, surjective for $n \geq 2$?

answer. No. Consider $n = 2$. Let

$$T = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then $T \in SL(2)$. Suppose $T \in \exp(A)$ where $A \in \mathfrak{sl}(2)$ is a traceless matrix. Then by Schur decomposition

$$U^{-1}AU = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

If $a = 0$ then

$$U^{-1}\exp(A)U = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

which has different spectrum than T which is not possible. If $a \neq 0$ then A is diagonalizable and so is $\exp(A)$. But T is not diagonalizable thus T is not in the image of the exponential map

If $B \in SL(n, \mathbb{R})$, is any matrix with some of the eigenvalues negative (and determinant 1) then $B \neq e^A$ for any A since e^A has all positive eigenvalues.

602. Give an counter-example to show that $\exp : \mathfrak{gl}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is not surjective.

answer. No. If $B \in GL(n, \mathbb{R}) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, and $e^A = B$ then eigenvalues λ_1, λ_2 of A satisfy $e^{\lambda_1} = -1$ and $e^{\lambda_2} = -2$ meaning $\lambda_1 = i\pi$ and $\lambda_2 = \ln 2 + i\pi$. Since A is real, its eigenvalues are complex conjugate. Contradiction.

2019-05-03

603. Reduce the general first order linear PDE $a(x,y)u_x + b(x,y)u_y = 0$ to a system of ODEs in x, y. What is the relation between the solution of the ODE and the PDE?

answer. The problem can be reduced to the ODE

$$\dot{x} = a(x,y), \qquad \dot{y} = b(x,y)$$

The solution of the PDE is a first integral of the above ODE since u is constant in the characteristic direction $(a(x,y), b(x,y))$, that is $u(x(t), y(t)) = c$.

The solution of the ODE generates the characteristic curves of the solution of the PDE.

604. Reduce the general first order linear PDE $a(x,y)u_x + b(x,y)u_y = 0$ to a first order exact ODE equation. What is the relation between the solution of the ODE and the PDE?

answer. The problem can be reduced to the ODE

$$\frac{dx}{a} = \frac{dy}{b}, \qquad b(x,y)dx - a(x,y)dy = 0$$

The solution of the PDE is constant on the integral curves of the ODE.

If $g(x,y) = c$ is an implicit solution of the ODE, that is if $g_x = b$, $g_y = -a$ then the general solution of the PDE is $u(x,y) = f(g(x,y))$ since

$$au_x + bu_y = af'g_x + bf'g_y = 0.$$

Trick. Think $x = t$, then

$$u_t + v \cdot \partial_y u = 0, \qquad v = \frac{b(t,y)}{a(t,y)}$$

which is the transport equation which is the first integral equation for the ODE $\frac{dy}{dt} = v$.

605. Solve $au_x + bu_y = c, a, b, c$ are constants.

answer. $u = f(ay - bx) + \frac{c}{a}x$. The first term is the homogeneous solution, the second one is non-homogeneous solution.

606. de Rham cohomology vs fundamental theorem of calculus?

answer. Every 1-form in 1D is automatically closed (since $d^2 = 0$) and exact by the FTC. de Rham cohomology, which (roughly speaking) measures precisely the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds.

607. Which implies the other: star-shaped, convex, contractible?

answer. convex implies star-shaped implies contractible.

608. If the divergence of a 2d-field is sign-definite in a region then no periodic orbits can lie in that region. Which result shows this? What is the idea of this result?

answer. This is Bendixon's criteria which can be shown via Green's theorem. The idea is if divergence of a v.f. is, say, positive in a region then the normal component of the v.f. can not vanish everywhere on its boundary. This can be generalized to Dulac's criteria.

609. Let $H_1 = H^2(0,1) \cap H_0^1(0,1)$, $H_{1/2} = H_0^1(0,1)$, $H = L^2(0,1)$. Then what can be said about the inclusions $H_1 \subset H_{1/2} \subset H$?

answer. They are dense and compact.

2019-05-01

610. A path-connected space whose fundamental group is trivial is called?

answer. simply-connected. The fundamental group of a topological space is an indicator of the failure for the space to be simply-connected.

611. Show that a contractible space is simply-connected.

answer. In a contractible space X, the identity map is homotopic to a null map $x \to x_0$ for some $x_0 \in X$. Hence any loop at x_0 is homotopic to the trivial loop and $\pi_1(X, x_0)$ is trivial. Moreover, since a contractible space is path-connected, the fundamental group based at any other point is also trivial.

612. For which topological spaces, the base point of the fundamental group makes no difference. Why?

answer. For path connected ones. The fundamental groups based at points x_1 and x_2 are isomorphic if there is a path p from x_1 to x_2. reason. If c is a loop at x_1 then pcp^{-1} is a loop around x_2. Show that the map $c \to pcp^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_2)$ is an isomorphism.

613. Define the fundamental group $\pi_1(X, x_0)$ of a topological space X with base point x_0. What is the group multiplication?

answer. It is the set of all loops modulo homotopy (that is homotopic loops are considered identical) with the group multiplication

$$(f * g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Thus the loop $f * g$ first follows the loop f with "twice the speed" and then follows g with "twice the speed".

614. Suppose we have a domain of $\mathbb{R}^n$ with the property that every closed 1-form is exact. Is this domain simply-connected?

answer. The answer is positive for n = 2 and negative for $n \ge 3$. Note that every closed 1-form on a simply-connected domain is exact. This is the converse question. There are domains which are not simply-connected but still all closed 1-forms are exact if $n \ge 3$. https://www.csun.edu/~vcmth02i/Forms.pdf

615. How is the derivative of a map $f : M \to N$ defined between manifolds?

answer. It is defined as the linear map $Df : TM \to TN$ which carries the tangent vector v at p defined by a curve γ to the tangent vector $Df(v)$ which is the tangent vector to the curve $f \circ \gamma$ at $f(p)$.

616. Express the differential of a map between manifolds in local coordinates?

answer. Let $f : M \to N$. Choose coordinates x near p on M and y near $f(p)$ on N. Then $F = y \circ f \circ x^{-1}$ is a map between Euclidean spaces. The differential of f can be represented as the Jacobian of F.

617. In 2D, write the vorticity formulation of the incompressible barotropic Euler equation without forcing.

answer. $\frac{D\omega}{Dt} = \omega_t + \mathbf{u} \cdot \nabla\omega = 0$.

618. Solve incompressible, isentropic 2D Euler's equation without forcing

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p, \qquad \nabla \cdot \mathbf{u} = 0$$

for velocity $\mathbf{u}$ in terms of the initial vorticity ω_0, the flow ϕ_t of the velocity, and K the Biot-Savart operator.

answer. $\mathbf{u} = K(\omega_0 \circ \phi^{-1})$. The vorticity formulation is $\frac{D\omega}{Dt} = \omega_t + \mathbf{u} \cdot \nabla\omega = 0$ which can be solved in terms of the motion as $\omega(\phi(a,t),t) = \omega(a,0) = \omega_0(a)$ or $\omega(x,t) = \omega_0(\phi^{-1}(x,t))$.

619. Express the motion as the integral of the velocity.

answer. $\phi(a,t) = a + \int_0^t v(\phi(a,s),s)ds$.

620. Assume $\frac{D\omega}{Dt} = \omega_t + \mathbf{u} \cdot \nabla\omega = 0$ where $\omega(x,t)$ is a scalar and $\mathbf{u}(x,t)$ is a vector. What is the the solution? What is the physical meaning?

answer. $\omega(x,t) = \omega_0 \circ \phi^{-1}(x,t)$ or $\omega(\phi(a,t),t) = \omega_0(a)$ where $\phi(a,t)$ is the flow of $\mathbf{u}$ and $\omega(x,0) = \omega_0(x)$. This means ω is constant along the flow of $\mathbf{u}$.

621. Assume $\frac{D\omega}{Dt} = \omega_t + \mathbf{u} \cdot \nabla\omega = 0$ where $\omega(x,t)$ is a scalar and $\mathbf{u}(x,t)$ is a vector. Show that $\omega(x,t) = \omega_0 \circ \phi^{-1}(x,t)$ is the unique solution of the IVP.

answer. Let ω be any solution. Define $W(a,t) = \omega(\phi(a,t),t)$. Then $\frac{\partial W}{\partial t} = \omega_t + \nabla\omega \cdot \mathbf{u} = 0$. Hence $\omega(\phi(a,t),t) = W(a,t) = W(a,0) = \omega_0(a)$.

This shows that if there is a solution it must be given by the formula. On the other hand, the given formula is a solution.

2019-04-30

622. Given the Eulerian velocity, how do you define the motion?

answer. Eulerian velocity $\to$ the motion is defined as the solution of the ODE, $\frac{\partial\phi(a,t)}{\partial t} = u(\phi(a,t),t)$ and $\phi(a,0) = a$. Hence $\phi(a,t) = a + \int_0^t u(\phi(a,s),s)ds$.

answer.

623. Given a motion ϕ, how do you define the Eulerian velocity?

answer. $\phi(a,t)$ motion $\to$ Lagrangian velocity $U(a,t) = \frac{\partial\phi(a,t)}{\partial t}$ $\to$ Eulerian velocity $u(x,t) = U(\phi^{-1}(x,t),t)$ or $u(x,t) = \frac{\partial\phi}{\partial t}(a,t)\Big|_{a=\phi^{-1}(x,t)}$.

To find the Eulerian velocity, first take the time derivative of the motion and then plug in the inverse motion to the Lagrangian variable.

624. Are these two expressions identical?

$$\frac{\partial\phi}{\partial t}(a,t)\Bigg|_{a=\phi^{-1}(x,t)} \qquad \frac{\partial}{\partial t}\phi(\phi^{-1}(x,t),t).$$

What is the latter one?

answer. They are not identical. Second one is $\frac{\partial}{\partial t}x = 0$.

625. What is the relation between the Lagrangian velocity U and Eulerian velocity u?

answer. $u(x,t) = U(\phi^{-1}(x,t),t)$ and $U(a,t) = u(\phi(a,t),t)$.

626. What is the Beale-Kato-Majda criterion?

answer. It is a necessary and sufficient condition for the global existence of smooth solution of the 3D incompressible Euler equations.

Beale, Kato, and Majda (1984) proved that a smooth solution of the 3D incompressible Euler equations breaks down on a time-interval $[0,T_*]$ if and only if

$$\int_0^T \|\omega\|_\infty(t)dt \to \infty \qquad \text{as } T \uparrow T_*$$

627. For the Euler Equations, what is known about the existence of global in time smooth - or weak - solutions in 2D and 3D for the initial value problem?

answer. 2D: there exist global smooth solutions. 3D: existence of global-in-time smooth — or weak — solutions. (see Hunter's notes)

628. Consider the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

Write a 1-form α so that if that form is exact then the system is Hamiltonian.

answer. $\alpha = -Q(x,y)dx + P(x,y)dy$.

Suppose

$$\alpha = dH = (\partial H/\partial x)dx + (\partial H/\partial y)dy$$

Then

$$\dot{x} = P = \frac{\partial H}{\partial y} \qquad \dot{y} = Q = -\frac{\partial H}{\partial x}$$

2019-04-29

629. Why Hamiltonian systems do not have attractors?

answer. Because of Liouville's theorem, which says that phase space volume is conserved by the flow of the system.

630. Why the conservation of mass equation (continuity equation) does not take diffusion into account?

answer. If a microscopic diffusion were to take place, we would not be able to tell because the molecular masses are identical, so we would not be able to tell state 1 from state 2.

631. Write the solution to $\Delta u = f$ in $V \subset \mathbb{R}^3$ with $u = g$ on ∂V as a sum of homogeneous and an EXPLICIT particular solutions.

answer.

$$u(x) = u_c(x) + u_p(x)$$

where

$$u_p(x) = \int_V \frac{f(x')}{-4\pi |x - x'|} d^3x'$$

$$\Delta u_c = 0, \text{ in } V, \qquad u_c = g(x) - u_p(x) \text{ on } \partial V$$

632. What is the Laplacian of the inverse distance in 3D?

answer. $-4\pi\delta^3(\vec{r} - \vec{r}_0)$

633. in 3D: $\Delta\left(\frac{1}{|\vec{r}-\vec{r}_0|}\right) = ?$

answer. $-4\pi\delta^3(\vec{r} - \vec{r}_0)$.

634. Show that the Fredholm alternative is implied by $KerA^T = (ImA)^\perp$.

Fredholm Alternative: Given A, b, exactly one of the following must hold: (1) $\exists x$, $Ax = b$, (2) $\exists y$, $y^T A = 0$, $y^T b \neq 0$.

answer. Suppose $Ax = b$ has no solution, then $b \notin Im(A)$ hence $b = Ax + y$ for some x and some non-zero $y \in (ImA)^\perp = KerA^T$. Thus $y^T b = (y^T A)x + y^T y = |y|^2 \neq 0$. Hence if (1) does not hold then (2) must hold. Also it is easy to see that if (1) holds then (2) can not hold.

2019-04-28

635. State Fredholm Alternative for matrices.

answer. Given A, b, exactly one of the following must hold: (1) $\exists x$, $Ax = b$, (2) $\exists y$, $y^T A = 0$, $y^T b \neq 0$.

Stated differently, either $Ax = b$ has a solution, or if not, the homogeneous adjoint problem $A^T y = 0$ has a non-trivial solution with $y^T b \neq 0$.

proof. Clearly both can not hold at the same time. This statement is equivalent to the statement $Ker A^T = (Im A)^{\perp}$. Because suppose $Ax = b$ has no solution, then $b \notin Im(A)$ hence $b = Ax + y$ for some x and some non-zero $y \in (Im A)^{\perp} = Ker A^T$. Thus $y^T b = (y^T A)x + y^T y = |y|^2 \neq 0$.

636. Let $\omega = \omega_{ij} dx^i \wedge dx^j$, $X = X_i \frac{\partial}{\partial X^i}$ and $Y = Y_i \frac{\partial}{\partial Y^i}$. Find $\omega(X, Y)$.

answer. $\omega(X, Y) = \omega_{ij}(X_i Y_j - Y_i X_j)$

637. Define the action of a differential 2-form on 2 tangent vectors in coordinates.

answer. $\omega(X, Y) = \omega_{ij}(X^i Y^j - X^j Y^i)$, where $\omega_{ij} = \omega(\partial_i, \partial_j)$, $X = X^i \partial_i$, $Y = Y^i \partial_i$.

638. Define the action of a differential 1-form on a tangent vector in coordinates.

answer. $\omega = \omega_i dx^i$ and $X = X_i \frac{\partial}{\partial x^i}$. Then $\omega(X) = \omega_i X_i$.

639. What are two compact forms of a Hamiltonian system?

answer. $z = (q, p) \in \mathbb{R}^{2n}$, (1) $\frac{dz}{dt} = J \nabla H(z)$ with $J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}$. (2) $\dot{z} = \{z, H\}$ which is to be understood as $\dot{q}^i = \left\{q^i, H\right\}$ and $\dot{p}^i = \left\{p^i, H\right\}$.

2019-04-27

640. Assume $\nabla \times \mathbf{u} = \mathbf{0}$ and decays sufficiently fast at infinity on $\mathbb{R}^3$. Find ψ which satisfies $\nabla \psi = \mathbf{u}$.

answer.

$$\psi(x) = \Delta^{-1}(\nabla \cdot \mathbf{u}) = \int_{\mathbb{R}^3} \frac{(x-y)}{4\pi|x-y|^3} \mathbf{u}(y) d^3 y$$

Proof. Integrate by parts the solution of $\Delta \psi = \nabla \cdot \mathbf{u}$, which is

$$\psi(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} (\nabla \cdot \mathbf{u}(y)) d^3 y$$

641. Give all the possible incompressible irrotational flows on $\mathbb{R}^3$ which decay at infinity.

answer. Must be zero. See [IIFDSF].

642. Prove that any incompressible irrotational flows on $\mathbb{R}^3$ which decay sufficiently fast at infinity is zero. [IIFDSF]

answer. Since $\nabla \times \mathbf{u} = \mathbf{0}$, we have $\nabla\psi = \mathbf{u}$ where

$$\psi(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|}(\nabla \cdot \mathbf{u}(y))d^3y$$

Since if $\nabla \cdot \mathbf{u} = 0$ then $\psi = 0$ hence $\mathbf{u} = 0$.

643. Find a function $G(x, y) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that $\Delta_x G(x, y) = 0$, if $x \neq y$ and $\int_{\partial B} \nabla_x G(x, y) \cdot \mathbf{n}(x) dS_x = -1$ for every ball centered around y.

answer. $G(x, y) = \frac{1}{4\pi|x-y|}$.

644. What is the growth rate for the integrability of the radial function $f(|x|)$ as $|x| \to \infty$ on $\mathbb{R}^n$?

answer. For some $\epsilon > 0$, $|f(|x|)| = O\left(\frac{1}{|x|^{n+\epsilon}}\right)$ as $|x| \to \infty$. A good way **to remember** is 1D case where this is the well known p-test for improper integrals.

To see, note that $\int_{\mathbb{R}^n} f(|x|)\mathrm{d}x = \omega_{n-1}\int_0^\infty f(r)r^{n-1}\mathrm{d}r$ where ω_n is the surface area of the of the n-sphere. The latter is integrable at infinity if for some $\epsilon > 0$, $|f(r)r^{n-1}| = O\left(\frac{1}{r^{1+\epsilon}}\right)$ as $r \to \infty$.

645. What is the integral of an exact 2-form on a sphere?

answer. $\int_{S^2} d\omega = \int_{\partial S^2} \omega = 0$ since $\partial S^2 = \emptyset$.

2019-04-26

646. Why is there no diffusion in the continuity equation but there is conservation of momentum equation?

answer. Do not know.

647. Show that for incompressible barotropic Euler equation

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = 0$$

with $\mathbf{u} \cdot \mathbf{n} = 0$ on the boundary, the kinetic energy is conserved.

answer. Take the inner product with $\mathbf{u}$, and use the identity

$$\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\,\mathbf{u}) = \frac{1}{2}\nabla\cdot\left(|\mathbf{u}|^2\mathbf{u}\right)$$

to obtain

$$\frac{d}{dt}\frac{1}{2}\int_{\Omega}|\mathbf{u}(x,t)|^2\,dx = 0$$

648. For a divergence-free vector field $\mathbf{u}$, find $\mathbf{A}$ such that

$$\nabla\cdot\mathbf{A} = \mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\,\mathbf{u})$$

answer. $\mathbf{A} = |\frac{1}{2}\mathbf{u}|^2\mathbf{u}$. Proof.

$$\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\,\mathbf{u}) = u_i\,(u_j\partial_j u_i) = u_j\partial_j u_i u_i = u_j\frac{1}{2}\partial_j(u_i^2) = \frac{1}{2}\mathbf{u}\cdot\nabla|\mathbf{u}|^2$$
$$= \nabla\cdot\left(|\frac{1}{2}\mathbf{u}|^2\mathbf{u}\right)$$

649. When is $\mathbf{u}\cdot\nabla f = \nabla\cdot(\mathbf{u}f)$ in general true?

answer. When $\mathbf{u}$ is a divergence-free vector field.

650. What is the physical meaning of Laplace equation?

answer. It describes the spatial distribution of a time-independent conserved quantity which is in equilibrium, that is its net flux is zero.

More details: When there is no advection, the flux is proportional to the gradient of the density u of that quantity, that is $F = -a\nabla u$. In integral form this reads $\int_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}dS = 0$ while in differential form it reads as the Laplace equation $\Delta u = 0$.

651. What does Laplace term in an equation correspond to?

answer. It describes the diffusion process. More details: The total flux through a boundary of a region V of a conserved quantity u is $\int_{\partial V}\mathbf{F}dS$. The diffusive part of the flux is $-d\nabla u$. By divergence theorem, the integral of the diffusive part becomes $-d\int_V \Delta u dV$.

652. Express the convection-diffusion equation for a conserved scalar c in a media with velocity $\mathbf{u}$, source S and non-homogeneous diffusion coefficient D.

answer. Flux is $\mathbf{F} = -D\nabla c + c\mathbf{u}$, sum of diffusion and convection/transport. Conservation law takes the form

$$c_t - \nabla\cdot(D\nabla c + c\mathbf{u}) = S$$

Special Case. When the velocity is divergence-free and the diffusion is homogeneous

$$c_t + \mathbf{u}\cdot\nabla c = D\Delta c + S$$

653. Express the heat equation as a conservation law in differential form

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = S$$

where U is the conserved state, $\mathbf{F}$ is its flux and S is a source/sink term.

answer.

$$T_t = d\Delta T + S,$$

$U = T$ is the temperature and $\mathbf{F}(U) = -d\nabla T$ (since the heat transport is from high to low) and S is source.

654. Express the continuity equation as a conservation law in differential form

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = S$$

where U is the conserved state, $\mathbf{F}$ is its flux and S is a source/sink term.

answer.

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

$U = \rho$ is density, $\mathbf{F}(U) = \rho\mathbf{u}$ is the mass transport by the the fluid flow and $S = 0$.

Question. The heat equation may include a convective flux, $\mathbf{F} = -d\nabla T + \mathbf{u}T$ which is the heat equation in Rayleigh-Benard convection. However, continuity equation does not contain a diffusion flux.

655. Express a conservation law in integral form and show how it can be written in differential form.

answer.

$$\frac{d}{dt}\int_\Omega U d\mathbf{x} + \int_{\partial\Omega} \mathbf{F}(U) \cdot \mathbf{n} ds = \int_\Omega S(U,t) d\mathbf{x}$$

where Ω is any fixed domain, U is the conserved state, $\mathbf{F}$ is the flux of the conserved state, $\mathbf{n}$ is the outward unit normal on the boundary $\partial\Omega$ and S is a source/sink term.

Using the divergence theorem, the differential form is

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = S$$

656. What is a nondegenerate critical point of a function from $f : \mathbb{R}^n \to \mathbb{R}$? What is a nondegenerate critical point of a vector field $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$?

answer. Function: $\nabla f(x) = 0$, $\det(D^2 f(x)) \neq 0$.

Vector field: $\mathbf{u}(x) = 0$, $\det(D\mathbf{u}(x)) \neq 0$.

2019-04-25

657. Show that the identity map on X is null-homotopic if and only if any continuous map on X is also null-homotopic.

answer. One way is trivial. For the other way, we know the existence of a homotopy $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$. Now $G(x,t) = H(f(x),t)$ is a homotopy between $f(x)$ and the constant map $x \to x_0$.

658. Prove asymptotical stability of an equilibrium when a strict Lyapunov function for that point exists. (strict = positive definite function with negative-definite time derivative along trajectories)

answer. Let $x(t)$ be a trajectory with $x(0) = x_0$ which lies in the domain U of Lyapunov function V. By the Lyapunov stability of the equilibrium it can be shown that the $\{x(t;x_0) : t \geq 0\}$ stays in U for all $x_0 \in V$ where V is a neighborhood of 0. Take any $x_0 \in V$. Then since $V(x(t))$ is strictly decreasing and bounded from below, it follows that $\lim_{t\to\infty} V(x(t)) = c$.

Suppose $c \neq 0 = V(x_e)$. Then $-\gamma = \max_{t\geq 0} \dot{V}(x(t)) < 0$ since the set $\{x(t) : t \geq 0\}$ is contained in a compact set $\{x : c \leq V(x) \leq V(x_0)\}$.

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \\ &\leq V(x(0)) - \gamma t \end{aligned}$$

which is a contradiction.

659. Prove Lyapunov stability of an equilibrium when a non-strict Lyapunov function L for that point exists. (non-strict = positive definite function with non-positive time derivative along trajectories)

answer. Take any closed spherical shell S centered at the equilibrium x. Then L being continous and S being compact, L has a positive minimum m on S. By continuity we can find a small open neighborhood N of x where $L(x) < m/2$ on N. Since L can not increase along trajectories, a trajectory starting from N can not enter the shell S.

2019-04-24

660. Show that if $A \in M_n(\mathbb{C})$ has an orthonormal basis of eigenvectors then there is a diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \cdot, \lambda_n)$ consisting of the eigenvalues of A and a unitary matrix $U = (u_1, u_2, \ldots, u_n)$ consisting of eigenvectors of U (unitary means $UU^* = U^*U = I$ where $*$ denotes conjugate transpose) such that $A = UDU^*$.

answer. If $\{u_1, u_2, \ldots, u_n\}$ is orthonormal then U is a unitary matrix such that

$$Ue_k = u_k, \qquad U^* u_k = e_k,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$. Note that the first equality is always true while the second one is a consequence of orthonormality.

$$U^* A U e_k = U^* A u_k = U^* \lambda_k u_k = \lambda_k e_k = D e_k$$

661. What is the relation between the vorticity and the velocity gradient?

answer. Let $R = \frac{1}{2}\left(\nabla v - \nabla v^T\right)$ be the anti-symmetric part of the velocity gradient. Then for $\omega = \nabla \times v$ and any vector h,

$$Rh = \frac{1}{2}\omega \times h$$

Biot-Savart

662. If the vorticity and the normal component on the boundary of an incompressible velocity field is specified in a bounded, simply-connected domain, then its ... is uniquely defined

answer. tangential component on the boundary.

663. (Biot-Savart) Let $D \subset \mathbb{R}^3$ be a bounded, simply-connected region. Given $\boldsymbol{\omega}$ in D and $\mathbf{g}$ on ∂D, show that there exists unique $\mathbf{u}$,

$$\nabla \times \mathbf{u} = \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{u} = 0, \text{ in } D, \qquad \mathbf{u} \times \mathbf{n} = \mathbf{g} \text{ on } \partial D$$

answer.

Existence. By the Biot-Savart integral we can always find

$$\nabla \times \mathbf{U} = \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{U} = 0, \qquad \text{in } \mathbb{R}^3.$$

Now show that the problem

$$\Delta f = 0 \text{ in } D, \qquad \nabla f \times \mathbf{n} = \mathbf{g} \text{ on } \partial D.$$

has a unique solution ??? ToDO

Then

$$\mathbf{u} = \mathbf{U} + \nabla f$$

satisfies all the required conditions.

Uniqueness. Let for $i = 1, 2$, $\nabla \cdot \mathbf{u}_i = 0$, (tangential component of the velocity) $\mathbf{u}_i \times \mathbf{n} = \mathbf{g}$ in ∂D and $\nabla \times \mathbf{u}_i = \omega$ in D. In a simply-connected domain $\mathbf{u}_1 - \mathbf{u}_2 = \nabla f$ so that $\Delta f = 0$ and $\nabla f \times \mathbf{n} = \mathbf{0}$. That is f is constant along the boundary. By Maximum principle for harmonic functions, f is constant in D so that $\mathbf{u}_1 = \mathbf{u}_2$. This means f is constant so that $\mathbf{u}_1 = \mathbf{u}_2$.

664. (Biot-Savart) Let $D \subset \mathbb{R}^3$ be a bounded, simply-connected region. Given $\boldsymbol{\omega}$ in D and g on ∂D satisfying

$$\int_{\partial D} g dS = 0$$

show that there exists unique $\mathbf{u}$,

$$\nabla \times \mathbf{u} = \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{u} = 0, \text{ in } D, \qquad \mathbf{u} \cdot \mathbf{n} = g \text{ on } \partial D$$

answer. Note: Simply-connectedness is only used in the uniqueness part. If not simply-connected, then is non-uniqueness possible?

Existence. By the Biot-Savart integral we can always find

$$\nabla \times \mathbf{u}_{\text{BS}} = \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{u}_{\text{BS}} = 0, \qquad \text{in } \mathbb{R}^3.$$

Now consider the Neumann problem

$$\Delta f = 0 \text{ in } D, \qquad \nabla f \cdot \mathbf{n} = g - \mathbf{u}_{\text{BS}} \cdot \mathbf{n} \text{ on } \partial D.$$

which has a unique solution f up to a constant since

$$\int_{\partial D} (g - \mathbf{u}_{\text{BS}} \cdot \mathbf{n}) dS = \int_{\partial D} g dS - \int_D \nabla \cdot \mathbf{u}_{\text{BS}} dV = 0$$

Then

$$\mathbf{u} = \mathbf{u}_{\text{BS}} + \nabla f$$

satisfies all the required conditions.

Uniqueness. Assume $\mathbf{u}_1$, $\mathbf{u}_2$ be two solutions. In a simply-connected domain, $\nabla \times (\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{0}$ so that $\mathbf{u}_1 - \mathbf{u}_2 = \nabla f$ and $\Delta f = 0$ and $\nabla f \cdot \mathbf{n} = 0$. By the Neumann problem for Laplace equation f is constant so that $\mathbf{u}_1 = \mathbf{u}_2$.

665. Under which conditions can the vorticity be uniquely inverted on bounded domains of 3-space?

answer. (Biot-Savart) When the domain is simply-connected and either the normal component or the tangential component of the velocity is prescribed at the boundary. See questions: item 663, 664.

666. State Biot-Savart formula on a domain with boundary in $\mathbb{R}^3$.

answer. ToDO

667. State Biot-Savart formula on whole $\mathbb{R}^3$.

answer. If $\boldsymbol{\omega}$ decays sufficiently fast at infinity and $\nabla \cdot \boldsymbol{\omega} = 0$ then

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{\boldsymbol{\omega}(\mathbf{y}) \times (\mathbf{x} - \mathbf{y})}{4\pi |\mathbf{x} - \mathbf{y}|^3} d\mathbf{y}$$

is the unique solution to

$$\nabla \times \mathbf{u} = \omega, \qquad \nabla \cdot \mathbf{u} = 0.$$

668. Assume that ω is divergence-free and decays sufficiently fast at infinity on $\mathbb{R}^3$. Find the vector potential $\mathbf{u}$ such that $\omega = \nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u} = 0$.

<u>`answer`</u>.

$$\mathbf{u}(x) = (-\Delta)^{-1}(\nabla \times \omega) = \int_{\mathbb{R}^3} \frac{\omega(y) \times (x-y)}{4\pi|x-y|^3} dV(y)$$

669. Assume that ω is divergence-free and decays sufficiently fast at infinity on $\mathbb{R}^3$. Show that $\omega = \nabla \times \mathbf{u}$ with $\nabla \cdot \mathbf{u} = 0$ where

$$\mathbf{u}(x) = (-\Delta)^{-1}(\nabla \times \omega) = \int_{\mathbb{R}^3} \frac{\omega(y) \times (x-y)}{4\pi|x-y|^3} dV(y)$$

Where do we use the decay condition? Where do we use that ω is divergence-free?

<u>`answer`</u>. *Step 1.* Taking the curl, $\nabla \times \omega = \nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \Delta \mathbf{u} = -\Delta \mathbf{u}$.

Step 2. By inverting the Laplace operator, we get

$$\mathbf{u}(x) = \int_{\mathbb{R}^3} G(x-y)\,(\nabla \times \omega(y))\, dV(y)$$

where $G(x-y) = \dfrac{1}{4\pi|x-y|}$ is the Green's Function.

Step 3. Using $\int_\Omega f \nabla \times \mathbf{g} dV = \int_{\partial\Omega} f\mathbf{n} \times \mathbf{g} dS - \int_\Omega \nabla f \times \mathbf{g} dV$, we have

$$\mathbf{u}(x) = \int_{\mathbb{R}^3} -\nabla_y G(x,y) \times \omega(y) dV(y)$$

Step 4. Noting $-\nabla_y G(x,y) = \dfrac{y-x}{4\pi|x-y|^3} = \nabla_x G(x,y)$, the integral representation follows.

Step 5. Now why is $\nabla \cdot \mathbf{u} = 0$? Direct computation.

Decay condition is used in two places: (1) TO ensure that the integrals converge, (2) The boundary integral in the integration by parts is canceled.

Divergence-free condition for the vorticity is a compatibility condition. But is it needed in the actual construction?

670. Show that the divergence of $\mathbf{u}(x) = \int_{\mathbb{R}^3} \frac{\omega(y)\times(x-y)}{4\pi|x-y|^3} dV(y)$ is zero.

<u>`answer`</u>. Use $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B})$ and the fact that curl of a radial vector field is zero.

Lie Groups and Algebras

671. What is the Lie bracket of matrix Lie algebras?

<u>`answer`</u>. Matrix commutator.

672. Under which operations are matrix Lie algebras closed?

<u>`answer`</u>. They are closed under matrix addition, scalar multiplication (since they form a vector space) and the matrix commutator.

673. Transpose of matrix commutator?

<u>`answer`</u>. $[A,B]^T = (AB - BA)^T = (B^T A^T - A^T B^T) = [B^T, A^T]$.

674. Verify that matrix commutator of skew-symmetric matrices is ...

<u>`answer`</u>. Skew symmetrix. $[A,B]^T = [B^T, A^T] = [-B, -A] = [B, A] = -[A, B]$.

675. What is the Lie algebra of $SU(n)$?

<u>`answer`</u>. $\mathfrak{u}(n)$ is the Lie algebra of skew-Hermitian matrices with zero-trace. If $I = e^A e^{A^*}$ means $A^* = -A$ and $\det e^A = 1$ implies trace of A is zero. Note that diagonal of skew-Hermitian matrices are purely imaginary and not necessarily zero.

676. What is the diagonal of skew Hermitian matrices?

<u>`answer`</u>. Purely imaginary.

677. What is Lie algebra of unitary group $U(n)$? Why?

<u>`answer`</u>. $\mathfrak{u}(n)$ is the Lie algebra of skew-Hermitian matrices, $A^* = -A$. If $e^A \in U(n)$, $I = e^A \left(e^A\right)^* = e^A e^{A^*} = e^{A+A^*}$, and $A + A^* = I$ that is A is skew-Hermitian.

678. What is $SU(n)$?

<u>`answer`</u>. Special unitary matrices which are unitary matrices with determinant 1.

679. What is $\det(A)$ for $A \in U(n)$?

<u>`answer`</u>. $\det(A) = e^{i\theta}$, $\theta \in \mathbb{R}$.

680. What is $U(n)$?

<u>`answer`</u>. Lie group of unitary matrices $UU^* = U^*U = I$, complex analog of orthogonal matrices.

681. What is the tangent space $T_1 O(n)$?

<u>`answer`</u>. Real skew-symmetric matrices.

682. Show that the tangent space $T_1O(n)$ is the real skew-symmetric matrices.

<u>answer</u>. Let $\gamma : (-\epsilon, \epsilon) \to O(n)$ be a curve with $\gamma(0) = I$. Then $0 = \frac{d}{dt}\gamma(t)^T\gamma(t) = \gamma'(t)^T\gamma(t) + \gamma(t)^T\gamma'(t)$. At $t = 0$, this reduces to $0 = v^T + v$.

683. Why are $\mathfrak{o}_n$ and $\mathfrak{so}_n$ equal?

<u>answer</u>. Because $SO(n)$ is the connected component of $O(n)$ containing identity.

684. Suppose there is a Poisson bracket defined on $C^\infty(M)$. Where is the Lie algebra here?

<u>answer</u>. The vector space $C^\infty(M)$ with the Poisson bracket as the Lie bracket is a Lie algebra. Recall that a Poisson bracket is (i) skew-symmetric, (ii) bilinear, satisfies (iii) Jacobi identity and (iv) Leibniz identity. (i), (ii), (iii) are enough to say that the Poisson bracket is a Lie bracket on the vector space $C^\infty(M)$.

685. Show, using the definition that the Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space at the identity, that the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ of $SL_n(\mathbb{R})$ (the matrix group with determinant 1) is the set of all matrices in $M_n(\mathbb{R})$ with trace zero.

<u>answer</u>.

Step 1. $\mathfrak{sl}_n(\mathbb{R}) = \{\gamma'(0) | \gamma : (-\epsilon, \epsilon) \to SL_n(\mathbb{R})$ is differentiable with $\gamma(0) = I\}$.

Step 2. $\gamma'(0) = \lim_{h\to 0} \frac{\gamma(h)-\gamma(0)}{h} \in M_n(\mathbb{R})$.

Step 3. If γ is such a curve then $\det(\gamma(t)) = 1$ for all t so that $\frac{d}{dt}\det(\gamma(t)) = 0$

Step 4. $\frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = \operatorname{trace}(\gamma'(0)) = 0$ by Jacobi's formula.

686. If $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ is differentiable and $\gamma(0) = I$ then find

$$\frac{d}{dt}\bigg|_{t=0} \det(\gamma(t))$$

<u>answer</u>. $\frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = \operatorname{trace}(\gamma'(0))$. By Jacobi's formula,

$$\frac{d}{dt}\bigg|_{t=0} \det(\gamma(t)) = \operatorname{trace}(\gamma'(0)\operatorname{adj}(\gamma(0))) = \operatorname{trace}(\gamma'(0)\operatorname{adj}(I))$$

where adjugate matrix is defined by $A\operatorname{adj}(A) = \det(A)I$ so that $\operatorname{adj}(I) = I$.

687. What is the definition of a Lie algebra of a Lie group G?

<u>answer</u>. The Lie algebra $\mathfrak{g}$ of G is the tangent space to G at the identity:

$$\mathfrak{g} = \{\gamma'(0) \mid \gamma : (-\epsilon, \epsilon) \to G \text{ is differentiable with } \gamma(0) = I\}$$

How is the Lie bracket defined? http://web.stanford.edu/~tonyfeng/222.pdf

688. What is the general definition Lie algebra?

answer. It is a vector space $\mathfrak{g}$ over some field with a bilinear, skew-symmetric binary operation $[\cdot,\cdot]$ which satisfies the Jacobi identity.

689. Show that the vector space $\mathbb{R}^3$ can be made a Lie algebra.

answer. $\mathfrak{g} = \mathbb{R}^3$ with the bracket $[x, y] = x \times y$.

690. How is the Lie algebra of a matrix Lie group defined via exponential map?

answer. If G is a matrix group then its Lie algebra is

$$\mathfrak{g} = \{X \in \mathrm{Mat}(n, \mathbb{C}) \mid \exp(tX) \in G, \quad \forall t \in \mathbb{R}\}$$

The Lie bracket is given by the matrix commutator.

691. Show that for vectors $\mathbf{a} = (a_1, a_2, a_3)^\top$ and $\mathbf{b} = (b_1, b_2, b_3)^\top$ there exists a skew symmetric matrix $[\mathbf{a}]_\times$ such that

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

Also

$$[\mathbf{a} \times \mathbf{b}]_\times = [\mathbf{a}]_\times [\mathbf{b}]_\times - [\mathbf{b}]_\times [\mathbf{a}]_\times$$

answer. Define

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

692. What is the commutator of two matrices? When do two matrices commute?

answer.

$$[A, B] = AB - BA$$

The matrices commute when their commutator is zero.

693. What are properties of commutator?

skew symmetric $[A, B] = -[B, A]$, bilinear, satisfies Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

694. Show that for a square matrix A, $x^T Ax = 0$ for all x iff A is skew symmetric.

answer. By definition $(u^T v = v^T u)$ $x^T Ax = (Ax)^T x = x^T A^T x$. If A is skew-symmetric then this implies $x^T Ax = -x^T Ax$ that is $x^T Ax = 0$.

For the converse, $0 = e_i^T A e_i = A_{ii}$ for all i and

$0 = (e_i+e_j)^T A(e_i+e_j) = (e_i+e_j)^T (A_i+A_j) = A_{ii}+A_{ij}+A_{ji}+A_{jj} = A_{ij}+A_{ji}$

for all i, j.

695. For a square matrix find Ae_i and $e_i^T Ae_j$.

<u>answer</u>. $Ae_i = A_i$, the i-th column and $e_i^T Ae_j = A_{ij}$.

<u>Example</u>

$$e_1^T Ae_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b = A_{12}$$

696. What is the group of volume and orientation preserving linear transformations of $\mathbb{R}^n$?

<u>answer</u>. $SL(n, \mathbb{R})$.

697. Define $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n)$, $SO(n)$ and $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{o}(n)$, $\mathfrak{so}(n)$.

<u>answer</u>.

- $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$,
- $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$,
- $O(n) = \left\{A \in M_n(\mathbb{R}) \mid A \text{ is orthogonal, i.e.} A^T A = AA^T = I\right\}$,
- $SO(n) = O(n) \cap SL(n, \mathbb{R})$,
- $\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$,
- $\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}$,
- $\mathfrak{o}(n) = \left\{A \in M_n(\mathbb{R}) \mid A \text{ is skew-symmetric, i.e.} A^T = -A\right\}$,
- $\mathfrak{so}(n) = \mathfrak{o}(n)$

698. What is the exponential of the real skew symmetric matrix $\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$?

<u>answer</u>. $e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$, a rotation matrix (with determinant $+1$). This means if $\dot{x} = Ax$ where A is a real skew-symmetric then $x(t)$ is the counterclockwise rotation around the origin.

699. If $\dot{x} = A(t)x$ where $A(t)$ is skew symmetric then what can be said about $x(t)$?

<u>answer</u>. $x(t)$ moves on a sphere, that is $\|x(t)\|$ is constant.

700. The solution of the simple harmonic oscillator $\ddot{x} = -kx$ lies on the circle in the phase plane. In general when does the solutions of $\dot{x} = A(t)x$ lie on a sphere.

<u>answer</u>. If $\dot{x} = A(t)x$ where $A(t)$ is skew symmetric then the trajectory of $x(t)$ lies on a sphere. See [SSHO]

SSHO . If $\dot{x} = A(t)x$ where $A(t)$ is real skew symmetric then show that $x(t)$ moves on a sphere, that is $\|x(t)\|$ is constant.

answer.

$$\frac{d}{2dt}(x \cdot x) = x \cdot \dot{x} = x \cdot A(t)x = (x, A(t)x) = 0$$

since $x_i A_{ij} x_j = x_j A_{ji} x_i = -x_j A_{ij} x_i$ implies $x_i A_{ij} x_j = 0$.

701. Show that when A is $n \times n$ skew-symmetric matrix, $\exp(A) \in SO(n)$.

answer. (a) $\exp(A)(\exp(A))^T = \exp(A)\exp(A^T) = \exp(A + A^T) = I$.
(b) $\det(\exp(A)) = \exp(\mathrm{tr}(A)) = 1$.

Helmholtz-Hodge Decomposition

702. State the exact formula of the Helmholtz-Hodge decomposition on whole $\mathbb{R}^3$ for sufficiently fast vanishing vector fields at infinity (so that the integrals are defined)

answer.

$$\mathbf{w} = \nabla p + \nabla \times \mathbf{r}$$

where

$$p(x) = -\frac{1}{4\pi}\int \frac{\mathrm{div}\,\mathbf{w}\,(x')}{|x - x'|}dx', \qquad \mathbf{r}(x) = \frac{1}{4\pi}\int \frac{\mathrm{curl}\,\mathbf{w}\,(x')}{|x - x'|}dx'$$

This shows that such vector fields are uniquely determined by their curl and divergence.

703. Discuss the uniqueness of the Helmholtz decomposition

$$\mathbf{w} = \mathbf{u} + \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \text{ in D}, \qquad \mathbf{u} \cdot \mathbf{n} = 0, \text{ on } \partial D$$

in non-simply connected regions and the existence of harmonic fields on that domain.

answer. ToDO. A related problem is:

Is the Helmholtz decomposition unique on a compact manifold without boundary?

Yes. Reason: On a compact domain without boundary (such as the surface of a sphere), there are no non-constant harmonic functions.

See the paper “The Helmholtz-Hodge Decomposition—A Survey”.

704. (Alternative decomposition for Helmholtz-Hodge decomposition) Show that any $\mathbf{u}$ defined on $D \subset \mathbb{R}^3$ can be represented as

$$\mathbf{u} = \nabla \times \mathbf{r} + \mathbf{p}$$

with $\nabla \times \mathbf{p} = 0$ in D and $\mathbf{n} \times \mathbf{p} = 0$ on ∂D.

answer. Taking curl,

$$\Delta \mathbf{r} = -\nabla \times \mathbf{u} \quad \text{in } D, \quad \text{with} \quad \mathbf{n} \times (\nabla \times \mathbf{r}) = \mathbf{n} \times \mathbf{u} \quad \text{on } \partial D$$

It can be shown that the equation has a unique solution (does not mean this decomposition is unique). The terms $\nabla \times \mathbf{r}$, $\mathbf{p}$ are orthogonal and finally by the orthogonality the decomposition is unique. See the paper "The Helmholtz-Hodge Decomposition—A Survey".

705. (Existence for Helmholtz-Hodge decomposition) Show that for $\mathbf{w}$ defined on $D \subset \mathbb{R}^3$, the decomposition

$$\mathbf{w} = \mathbf{u} + \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \text{ in D}, \qquad \mathbf{u} \cdot \mathbf{n} = 0, \text{ on } \partial D$$

exists.

answer. We get the Neumann problem

$$\Delta p = \operatorname{div} \mathbf{w} \quad \text{in } D, \quad \text{with} \quad \frac{\partial p}{\partial n} = \mathbf{w} \cdot \mathbf{n} \quad \text{on } \partial D$$

Since the data satisfies the compatability condition $\int_D \nabla \cdot \mathbf{w} dV = \int_{\partial D} \mathbf{w} \cdot n$, the solution to this problem exists and is unique up to the addition of a constant to p. With this choice of p, define $\mathbf{u} = \mathbf{w} - \nabla p$. Hence such a decomposition exists.

Remarks.

(1) Now consider any such decomposition. Then we get the exact same Neumann problem for p so that ∇p and thus $\mathbf{u}$ are unique.

(2) The decomposition can easily be generalized to the non-homogeneous boundary data g with $\int_{\partial D} g dS = 0$, such that

$$\mathbf{w} = \mathbf{u} + \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \text{ in D}, \qquad \mathbf{u} \cdot \mathbf{n} = g, \text{ on } \partial D$$

706. (Proof of uniqueness for Helmholtz-Hodge decomposition) Show that for $\mathbf{w}$ defined on $D \subset \mathbb{R}^3$, the decomposition $\mathbf{w} = \mathbf{u} + \nabla p$ with $\mathbf{u}$ div-free and $\mathbf{u} \cdot \mathbf{n} = 0$ is unique using the orthogonality relation.

answer. We have $\int_D \mathbf{u} \cdot \nabla p dV = 0$. Suppose that $\mathbf{w} = \mathbf{u}_1 + \nabla p_1 = \mathbf{u}_2 + \nabla p_2$. Then $0 = \mathbf{u}_1 - \mathbf{u}_2 + \nabla (p_1 - p_2)$. Taking the inner product with $\mathbf{u}_1 - \mathbf{u}_2$ and integrating, we get

$$0 = \int_D \left\{ \|\mathbf{u}_1 - \mathbf{u}_2\|^2 + (\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla (p_1 - p_2) \right\} dV = \int_D \|\mathbf{u}_1 - \mathbf{u}_2\|^2 dV$$

which shows that $\mathbf{u}_1 = \mathbf{u}_2$ and $\nabla p_1 = \nabla p_2$.

707. (L^2 orthogonality for Helmholtz-Hodge decomposition) Show that if

$$\nabla \cdot \mathbf{u} = 0 \text{ in } D, \qquad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D$$

then

$$\int_D \mathbf{u} \cdot \nabla p dV = 0, \qquad \text{for al smooth } p.$$

<u>`answer`</u>. This is a simple consequence of divergence theorem and the fact that $\mathbf{u}$ is div-free.

$$\int_D \mathbf{u} \cdot \nabla p dV = \int_D \operatorname{div}(p\mathbf{u}) dV = \int_{\partial D} p\mathbf{u} \cdot \mathbf{n} dA = 0$$

708. State 2-term Helmholtz-Hodge Decomposition in a <u>contractible</u> domain with smooth boundary.

<u>`answer`</u>. A vector field w on D can be uniquely decomposed in the form

$$\mathbf{w} = \nabla \times \mathbf{u} + \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } D, \qquad \nabla \times \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D.$$

Remark: $\nabla \times \mathbf{u} \cdot \mathbf{n} = 0$ on ∂D means that circulation of $\mathbf{u}$ is zero on any closed curve on the boundary.

709. State 2-term Helmholtz-Hodge Decomposition on a domain with smooth boundary.

<u>`answer`</u>. A vector field w on D can be uniquely decomposed in the form

$$\mathbf{w} = \mathbf{u} + \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } D, \qquad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D$$

that is $\mathbf{u}$ is parallel to ∂D.

For proof, see Chorin-Marsden pg 37.

710. The motion of a fluid $\mathbf{u}$ in an infinite space $\mathbf{R}^3$ such that it vanishes at infinity is determinate when we know the values of the divergence $\theta(x) = \nabla \cdot \mathbf{u}(x)$ and the curl $\omega(x) = \nabla \times \mathbf{u}$. On the other hand, if the motion of the fluid is limited to a simply-connected region $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$, it is determinate if $\theta(x)$, $\omega(x)$ and the value of the flow normal to the boundary, $\mathbf{u}_n = \mathbf{u} \cdot \mathbf{n}$ for $x \in \partial\Omega$ are known.

711. Explain the terms in three-component Helmholtz-Hodge decomposition.

<u>`answer`</u>. (1) gradient term/irrotational part: expansion or contraction in three orthogonal directions, (2) curl term/divergence-free part: rotation about an instantaneous axis, and (3) irrotational and divergence-free part: translation.

712. Describe the elements of 3-term Helmholtz-Hodge decomposition of a smooth vector field.

answer. A smooth vector field on a domain can be uniquely decomposed into three components: 1. an irrotational component d, 2. an incompressible component r, 3. an harmonic component h, representing the translation part. $v = \nabla D + \nabla \times R + h = d + r + h$

The harmonic part being at the same time irrotational and incompressible can be put together with the first or the second term, giving a version of the decomposition with only two parts.