Dynamical Systems of ODE's

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Please let me know of any mistakes in these notes.

Contents

Intro

In this lecture, we will study systems of the form

$$\dot{x}(t) = \frac{dx(t)}{dt} = f(x(t)) \tag{1}$$

We call the variable $t \in \mathbb{R}$ as **time**, the vector valued function $x = x(t) \in \mathbb{R}^n$ the **position** (of a particle). We call $\dot{x}$ as the **velocity** (of that particle). The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is the **velocity field** and is a given smooth function.

(1) tells that the particle has no choice but to follow the direction of the velocity field. Our goal will be to determine "the behavior" of the position in time depending possibly on the initial position

$$x(0) = x_0$$

The existence and uniqueness theorem of the differential equations guarantees that the position $x(t)$ can be uniquely determined given an initial position

Chapter 2. Flows on the line

2.1. A Geometric Way of Thinking

Let

$$f : A \subset \mathbb{R} \to \mathbb{R}, \qquad x : I \subset \mathbb{R} \to A \tag{2}$$

We will study

$$\dot{x}(t) = \frac{dx(t)}{dt} = f(x(t)),$$

Consider

$$\dot{x} = \sin x, \qquad x(0) = x_0$$

We can solve it.

$$\frac{dx}{\sin x} = dt \implies t = \int \csc x dx = -\ln|\csc x + \cot x| + C$$

The solution is

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

Not very useful.

Listing 1: Code for Figure 2

```
s[x0_] := NDSolve[x'[t] == Sin[x[t]] && x[0] == x0, x, {t, 0, 10}]
sol = Evaluate@Table[x[t] /. s[x0], {x0, -5 Pi/4, 5 Pi/4, Pi/4}];
Plot[sol, {t, 0, 10}, PlotRange -> All, TicksStyle -> Directive[18]]
```

Question: For $x_0 = \frac{\pi}{4}$, $\lim_{t\to\infty} x(t) =?$

Geometric idea: t= time, $x(t)$ = position at time t, $\dot{x}$ = velocity at time t of an imaginary particle.

The particle moves to right if $f(x) = \dot{x} > 0$ and to left if $f(x) = \dot{x} < 0$.

The particle stays if $f(x) = \dot{x} = 0$.

A point x^* is called a **fixed point** of the system $\dot{x} = f(x)$ if $f(x^*) = 0$. Equivalently $x_e(t) = x^*$ is a solution, called an an **equilibrium solution**, of $\dot{x} = f(x)$, $x(0) = x^*$.

Example 1.

$$\dot{x} = \sin x$$

Sketch the solutions in the $x - t$ plane.

Solution. $\ddot{x} = \cos x \dot{x} = \cos x \sin x$. *For* $0 < x < \pi$, $\dot{x} > 0$.

For $0 < x < \pi/2$, $\ddot{x} > 0$.

For $\pi/2 < x < \pi$, $\ddot{x} < 0$.

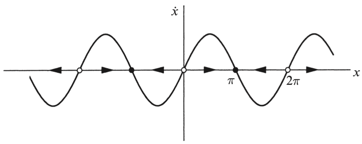


Figure 1: Phase portrait of $\dot{x} = \sin x$.

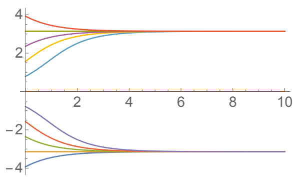


Figure 2: The solutions of $\dot{x} = \sin x$ for different initial codes. See Listing 1.

Definition 1. *A fixed point x^* is called an **(locally asymptotically) stable fixed point** if any solution with initial condition near x^* tends to x^* as $t \to \infty$. Otherwise it is called an **unstable fixed point**.*

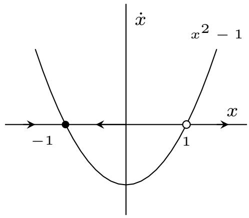
Definition 2. *If x^* is a stable fixed point then the set $B(x^*)$ consisting of all initial conditions for which the corresponding solution tends to x^* as $t \to \infty$ is called the **basin of attraction** of x^*.*

2.2. Fixed Point and Stability

Example 2. *Find all fixed points of*

$$\dot{x} = x^2 - 1$$

Sketch the phase portrait and classify the stability of equilibria. Find the basin of attraction of the stable equilibrium.



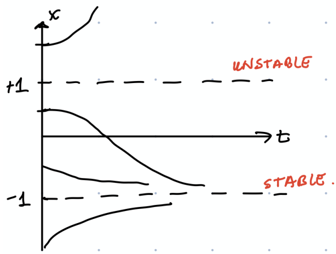


Figure 3: The solutions of $\dot{x} = x^2 - 1$.

The basin of attraction of the stable equilibrium $x = -1$ *is* $(-\infty, 1)$.

Example 3. *Sketch the phase portrait of*

$$\dot{x} = x - \cos x$$

and determine the stability of all fixed points. Hint: sketch $y = x$ *and* $y = \cos x$ *separately.*

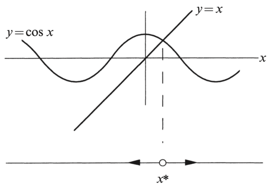


Figure 4: $f(x) = x - \cos x$

Example 4. *Consider*

$$\dot{x} = ax \tag{3}$$

Show that $x^* = 0$ is a stable fixed point if $a < 0$ and an unstable fixed point if $a > 0$.

Example 5. *Find and classify the fixed points of $\dot{x} = \sin x$.*

There are infinitely many fixed points: $x = k\pi$, $k \in \mathbb{Z}$. The fixed points $x = 2k\pi$ are unstable, $x = (2k+1)\pi$ are stable. See Figure 1 and Figure 5.

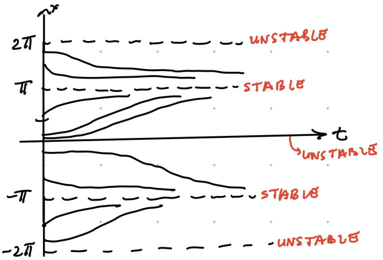


Figure 5: The solutions of $\dot{x} = \sin x$.

Example 6. *Sketch the solutions of $\dot{x} = x^2(6-x)$, $x(0) = x_0$, for $x_0 = 0, 1, 10$ in the $t - x$ plane.*

Fixed points are $x = 0, 6$.

$$\dot{x}: \qquad \uparrow (x=0) \uparrow (x=6) \downarrow$$

So $x = 0$ is half-stable and $x = 6$ is stable.

$$\ddot{x} = 3x^3(4-x)(6-x)$$

So inflection points are $x = 0, 4, 6$.

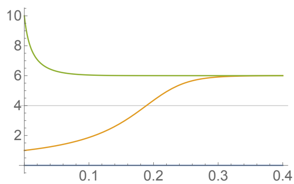


Figure 6: The solutions of $\dot{x} = x^2(6-x)$, $x(0) = x_0$, for $x_0 = 0, 1, 10$.

2.3. Population Growth

Let $N(t)$ be the population of a species at time t, then the rate of change

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}$$

Listing 2: Code for Figure 6

```
T = .4;
s[x0_] :=
 NDSolve[x'[t] == x[t]^2 (6 − x[t]) && x[0] == x0, x, {t, 0, T}]
sol = Evaluate@Table[x[t] /. s[x0], {x0, {0, 1, 10}}];
Plot[sol, {t, 0, T}, PlotRange −> All, TicksStyle −> Directive[18], Gric
```

is a conservation equation for the population.

The simplest model has no migration and the birth and death terms are proportional to N. That is

$$\frac{dN}{dt} = bN - dN$$

Setting $r = b - d$,

$$\dot{N} = rN, \qquad N(0) = N_0$$

The solution is $N(t) = e^{rt}N_0$. If $r > 0$, then the solution grows exponentially. This is not realistic, no species can grow indefinitely.

If $r < 0$, then the solution decays to zero exponentially.

A more realistic population growth model is called the **logistic equation**

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \tag{4}$$

$K > 0$ is a constant, called the carrying capacity. The growth rate is $r\left(1 - \frac{N}{K}\right)$ which decreases linearly with N and is negative for $N > K$.

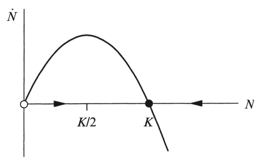


Figure 7: $f(N) = rN\left(1 - \frac{N}{K}\right)$

Picturing the solutions. Note that $\dot{N}$ increases with N when $0 < N < K/2$ and decreases when $N > K/2$. The solution is concave up, when $0 < N < K/2$ and concave down when $N > K/2$.

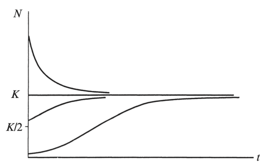


Figure 8: The solutions of $\dot{x} = rN\left(1 - \frac{N}{K}\right)$ with different initial conditions.

Listing 3: Code for Figure 10

```
s[x0_] :=
 NDSolve[x'[t] == -x[t]*Log[2 x[t]] && x[0] == x0, x, {t, 0, 10}]
sol = Evaluate@Table[x[t] /. s[x0], {x0, 0.01, 2, .2}];
Plot[sol, {t, 0, 10}, PlotRange -> All, TicksStyle -> Directive[18]]
```

The logistic growth model gives good results for the growth of simple organisms such as bacteria under ideal conditions. However, for more complex species, it fails.

Example 7.

$$\dot{N} = -aN\ln(bN), \qquad a, b > 0.$$

The fixed points are $N = 0$, $N = \frac{1}{b}$. *A graphical analysis shows that the* $N = 0$ *is unstable and* $N = 1/b$ *is stable.*

```
Plot[-n*Log[2 n], {n, 0, .8}, TicksStyle -> Directive[18]]
```

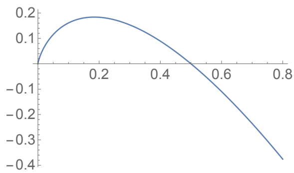


Figure 9: The plot of $f(N) = -aN\ln(bN)$, for $a = 1$, $b = 2$.

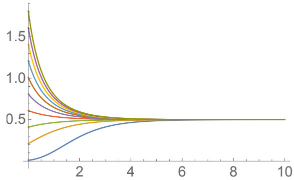


Figure 10: The solutions of $\dot{N} = -aN\ln(bN)$, for $a = 1$, $b = 2$ with different initial conditions. See Listing 3.

2.4. Linear Stability Analysis

Let x^* be a fixed point of (2), that is $f(x^*) = 0$. *Then* x^* *is stable if* $f'(x^*) < 0$, *unstable if* $f'(x^*) > 0$.

Proof. Let $x(t)$ be any solution which stays "sufficiently close" to x^* for all t.

$$\eta(t) = x(t) - x^*$$

Then we assume that $\eta(t)$ is small for all t. If $\lim_{t\to\infty} \eta(t) = 0$ then x^* is stable, and unstable otherwise (why?).

Taylor's expansion:

$$\dot{\eta} = \dot{x} = f(x^* + \eta) = f(x^*) + f'(x^*)\eta + \frac{f''(x^*)}{2!}\eta^2 + \cdots = f'(x^*)\eta + O(\eta^2)$$

since $f(x^*) = 0$. Here $O(\eta^2)$ denotes the remaining terms with η^k, $k \geq 2$. Now if $f'(x^*) \neq 0$ and if η is "small enough" then $O(\eta^2)$ part can be ignored:

$$\dot{\eta} \approx f'(x^*)\eta \tag{5}$$

This equation is called the **linearization** of (2) at the fixed point $x = x^*$.

Since $\eta(t)$ is small, $\eta(t) \approx \exp(f'(x^*)t)C$.

$$\lim_{t\to\infty} \eta(t) = 0, \qquad \text{if } f'(x^*) < 0$$

When $f'(x^*) > 0$, as t increases η becomes larger and $\lim_{t\to\infty} \eta(t) \neq 0$. (Note: when $f'(x^*) > 0$, $\eta(t) \approx \exp(f'(x^*)t)C$ is no lorger valid for large t as ignoring the $O(\eta^2)$ terms is not valid when η is not small.) □

Example 8. *Logistic equation: $f(N) = rN(1 - N/K)$ with fixed points $N^* = 0$ and $N^* = K$. Then $f'(0) = r > 0$, so that $N^* = 0$ is unstable and $f'(K) = -r < 0$ so that $N^* = K$ is stable.*

Example 9. *What can be said about stability of a fixed point when $f'(x^*) = 0$?*

Nothing. Consider

(a) $\dot{x} = -x^3$, *(b)* $\dot{x} = x^3$, *(c)* $\dot{x} = x^2$, *(d)* $\dot{x} = 0$,

All these systems have a fixed point $x^* = 0$ with $f'(0) = 0$. However the stability is different in each case. In (c) $x^* = 0$ is called ***half-stable****. In (d), the x-axis is a line of fixed points. Each fixed point is marginally stable, the perturbations do not decay to zero, but they do not grow either.*

Such examples seem artifical but they arise naturally in the context of bifurcations.

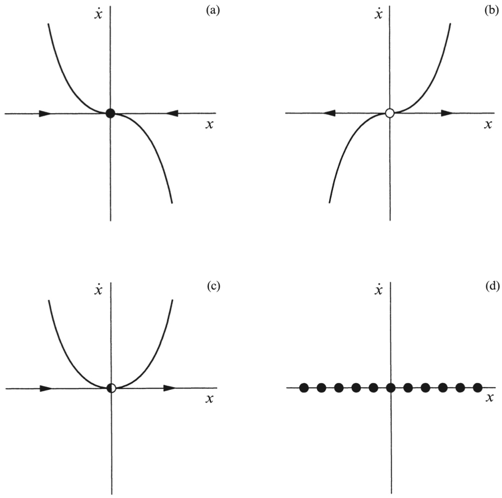


Figure 11: The graphs of (a)$-x^3$, (b) x^3, (c) x^2, (d) 0.

2.5. Existence and Uniqueness Theory

Example 10. *Show that the solution to $\dot{x} = x^{1/3}$ with $x(0) = 0$ is not unique.*

One solution is $x(t) = 0$. Another solution is

$$\int x^{-1/3} dx = \int dt$$

which gives $\frac{3}{2}x^{2/3} = t + C$. The initial condition gives $C = 0$. Hence $x(t) = \left(\frac{2}{3}t\right)^{3/2}$ is another solution.

When uniqueness fails, our geometric approach fails. The system can not choose how to evolve.

Problem. Show that the above system has infinitely many solutions. **Answer**. In fact

$$x(t) = \begin{cases} 0, & t \leq c \\ \left(\frac{2}{3}(t-c)^{3/2}\right), & t > c \end{cases}$$

is a solution for each $c \geq 0$.

The reason of non-uniqueness: the slope $f'(0) = \infty$, the fixed point 0 is very unstable.

Existence and uniqueness theorem. If f and f' are continuous then the IVP (initial value problem)

$$\dot{x} = f(x), \qquad x(0) = x_0$$

has a unique solution $x(t)$ defined on some interval $(-\tau, \tau)$.

Remark. The continuity condition of f, f' can be replaced by the weaker condition of f being Lipschitz continuous.

Example 11. *Consider the IVP*

$$\dot{x} = 1 + x^2, \qquad x(0) = 0$$

By the above theorem, a unique solution exists. We can find the solution by separation of variables:

$$\int \frac{dx}{1+x^2} dx = \int dt \implies \arctan x = t + C$$

*By the initial condition, $x(t) = \tan t$. The solution exists only for $-\pi/2 < t < \pi/2$ as $x(t) \to \pm\infty$ as $t \to \pm\pi/2$. There is no differentiable function defined on a larger interval which solves the IVP. This phenemenon is called **finite time blow-up**.*

2.6. Impossibility of Oscillations

In all our examples so far, all trajectories either approached a fixed point, or diverged to $\pm\infty$. In fact, those are the only things that can happen for a vector field on the real line.

The solutions can not oscillate in 1D systems.

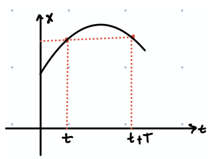


Figure 12: This can not be the solution of an autonomous 1D differential equation $\dot{x} = f(x)$.

Theorem 1. *Suppose $x(t)$ is a solution of $\dot{x} = f(x)$. Then it is not possible that $x(t+T) = x(t)$ for some t and $T > 0$ while $x(t+s) \neq x(t)$ for $0 < s < T$.*

Proof. Suppose on the contrary $x(t) = x(t+T)$ for some $T > 0$ and $x(t) \neq x(t+s)$ for all $0 < s < T$. Suppose $F' = f$ (such F always exists when f is continuous)

$$\begin{aligned} 0 &< \int_t^{t+T} \left(\frac{dx(s)}{ds}\right)^2 ds = \int_t^{t+T} f(x(s)) \frac{dx(s)}{ds} ds \\ &= \int_t^{t+T} \frac{d}{ds}\left(F(x(s))\right) ds = F(x(t+T)) - F(x(t)) = 0 \end{aligned}$$

□

Since oscillations are not possible, the only possible solutions of 1D autonomous ODE are

- *constant (equilibrium) solutions,*
- *solutions that monotonically approach an equilibrium solution,*
- *solutions that monotonically tend to $\pm\infty$ in finite/infinite time.*

2.7. Potentials

A 1D autonomous system

$$\dot{x} = f(x) \tag{6}$$

can always be put into gradient form

$$\dot{x} = -\frac{dV}{dx}. \tag{7}$$

by letting

$$V(x) = -\int f(x)dx$$

The function $V(x)$ is called the **potential function** of the system (6).

Lemma 1. *For any solution $x(t)$, $V(x(t))$ is non-increasing function of time t. $V(x(t))$ is a strictly decreasing function of t if $x(t)$ is not an equilibrium solution.*

Proof. If $x(t)$ is a solution then

$$\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\dot{x} = -\dot{x}^2 \leq 0$$

□

A conclusion of the above lemma is the following.

Theorem 2. *Suppose that the domain of f is $\mathbb{R}$ and f is a smooth function. Then the local minima of V are stable equilibrium solutions of (6) and the local maxima of V are unstable equilibrium solutions of (6).*

Proof. Since the domain of f has no boundary and f' is defined everywhere by smoothness, at a local extremum point x^* of V, $\frac{dV(x^*)}{dx} = 0$. By (7), x^* is an equilibrium solution of (6). □

Example 12.

$$\dot{x} = -x$$

Then $V(x) = -\int -xdx = x^2$. (choose $C = 0$).

<u>1st viewpoint</u>. For any initial condition $x(0) = x_0$,

$$\lim_{t\to\infty} x(t, x_0) = 0 = \min_{x\in\mathbb{R}} V(x)$$

That is any solution tends to the fixed point which is the minima of V which is $x = 0$.

<u>2nd viewpoint</u>. The graph analysis of $y = f(x)$ shows that $x = 0$ is a stable equilibrium.

<u>3rd viewpoint</u>. Since $f'(x) = -1$, linear stability analysis says that $x = 0$ is a stable equilibrium point.

Example 13. *A **bistable** system: $\dot{x} = x - x^3$. This time $V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ known as double well potential. There are two stable fixed points ± 1 which are minima of V and an unstable fixed point 0 which a maxima of V.*

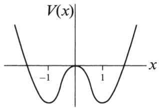


Figure 13: $V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

Chapter 2 Homework

- 2.2: 1, 3, 7, 8, 9
- 2.3: 1 (Solve the logistic equation (4)), 3, 4
- 2.4: 1, 3, 7, 8
- 2.5: 3, 4
- 2.6: 2
- 2.7: 1, 6

Chapter 3. Bifurcations

The qualitative structure of the flow can change as parameters are varied such as fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called **bifurcations**, and the points (x^*, r^*) with different types of dynamical behaviors in each neighborhood are called (local) **bifurcation points**.

3.1. Saddle-Node Bifurcation

In a saddle-node bifurcation, as a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

$$\dot{x} = r + x^2 \tag{8}$$

where $r \in \mathbb{R}$ is a parameter. (8) is the prototypical example of a saddle-node bifurcation.

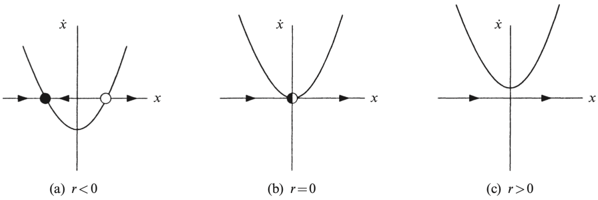
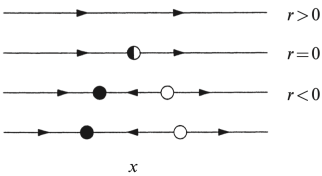


Figure 14: Fixed points and their stability for (8). $(x, r) = (0, 0)$ is a bifurcation point.



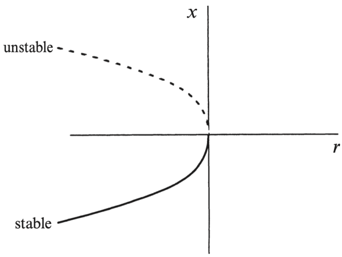
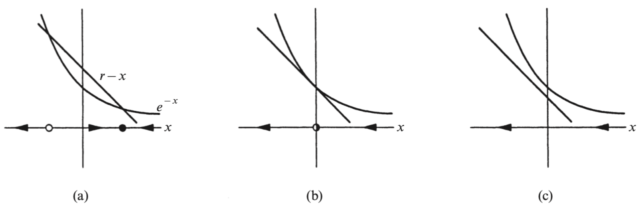


Figure 15: Bifurcation diagram of (8). This is the graph of equilibrium solutions of $\dot{x} = r + x^2$, that is $r = -x^2$. The solution curve $x = \sqrt{-r}$ is unstable while $x = -\sqrt{-r}$ is stable.

Example 14. *Give the linear stability analysis of (8).*

Solution. There are two fixed points $x^* = \pm\sqrt{-r}$ for $r < 0$, 1 for $r = 0$ and none for $r > 0$. For $r < 0$, $f'(\sqrt{-r}) = 2\sqrt{-r} > 0$ and $x^* = -\sqrt{-r}$ is unstable, $f'(-\sqrt{-r}) = -2\sqrt{-r} < 0$ and $x^* = -\sqrt{-r}$ is stable.

Example 15. *Show that $\dot{x} = r - x - e^{-x}$ undergoes a saddle-node bifurcation as r is varied. Find the bifurcation point.*



At the bifurcation point, $y = r - x$ and $y = e^x$ become tangent.

$$e^{-x} = r - x, \qquad \text{and} \qquad \frac{d}{dx}e^{-x} = \frac{d}{dx}(r - x)$$

From the second equation, $e^{-x} = 1$ so $x = 0$. The first equation yields $r = 1$. The bifurcation point $(x, r) = (0, 1)$.

Theorem 3 (No Bifurcation Theorem)**.** *Suppose*

$$f(x^*, r^*) = 0.$$

If

$$\frac{\partial f}{\partial x}(x^*, r^*) \neq 0$$

then the equation

$$f(x, r) = 0$$

has a unique solution $x = g(r)$ for each r in a sufficiently small neighborhood of r^*.

Proof. This is basically the statement of the implicit function theorem. □

Example 16. *Show that $\dot{x} = r - x - e^x$ has NO bifurcation as r is varied.*

This is because $f_x(x, r) = -1 - e^x < 0$. It can be seen easily that the system has always a stable fixed point.

3.2. Transcritical Bifurcation

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, for population models, zero equilibrium is always a solution. However, such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2 \tag{9}$$

There is a fixed point at $x^* = 0$ for all values of r.

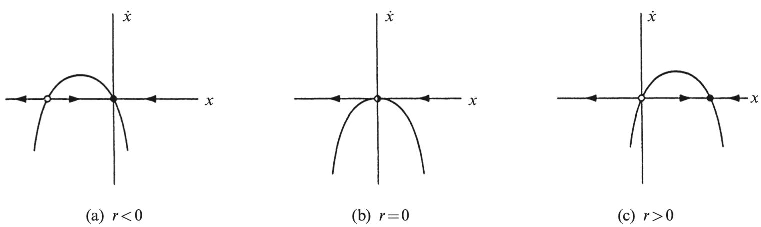


Figure 16: Fixed points and their stability for (9). $(x, r) = (0, 0)$ is a bifurcation point.

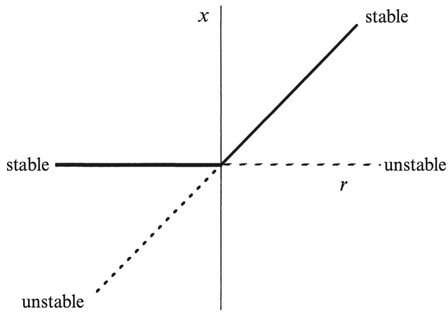


Figure 17: Bifurcation diagram for (9). This is also known as **exchange of stabilities**.

3.4. Pitchfork Bifurcation

This bifurcation is common in physical problems that have a symmetry. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs.

Two types of pitchfork bifurcation.

Supercritical pitchfork bifurcation.

$$\dot{x} = rx - x^3 \tag{10}$$

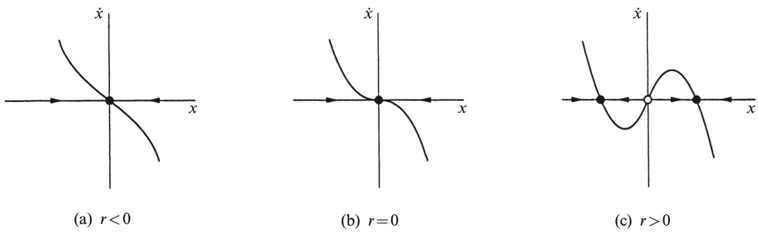


Figure 18: Phase portrait for (10). $x = 0$ is always a fixed point. $(x, r) = (0, 0)$ is a bifurcation point. For $r < 0$ the origin is the only fixed point which is stable. For $r > 0$, $x = 0$ is unstable, and there are two other fixed point $x = \pm\sqrt{r}$ which are stable.

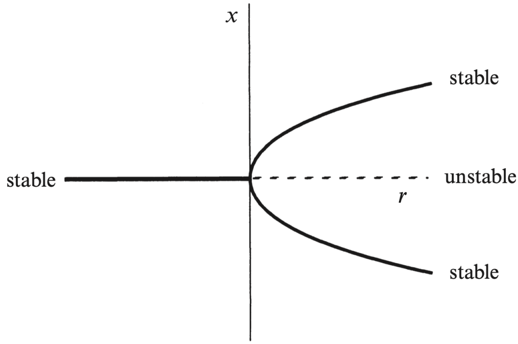


Figure 19: Bifurcation diagram for (10).

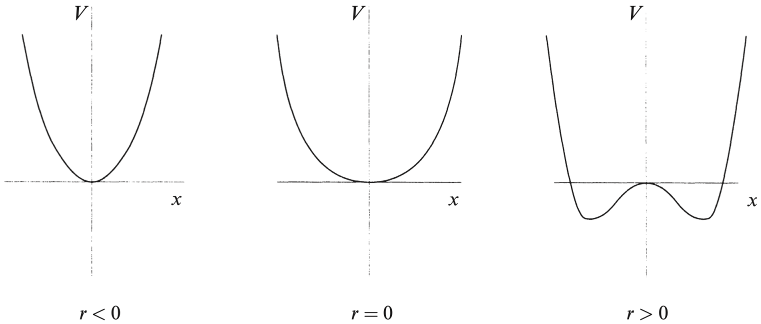


Figure 20: The potential function $V(x) = \frac{-1}{2}rx^2 + \frac{1}{4}x^4$ of the equation (10). When $r < 0$, $x = 0$, is a minimum of V. When $r > 0$, $x = \pm\sqrt{r}$ are minima of V.

Example 17. *Analyze the* ***subcritical pitchfork bifurcation***

$$\dot{x} = rx + x^3.$$

3.7. Insect Outbreak

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N)$$

- $N(t)$ is the budworm population.
- K is the carrying capacity.
- R is the birth rate of the budworms.
- $p(N)$ is the predation term.

There are different ways of choosing the predation term. Here

$$p(N) = \frac{BN^2}{A^2 + N^2}.$$

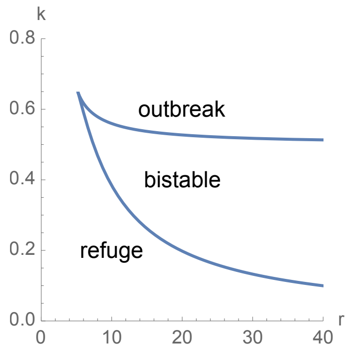


Figure 21: Insect outbreak bifurcation diagram. See Code 1.

Code 1.

```
r = (2 x^3)/(x^2 - 1);
k = (2 x^3)/(1 + x^2)^2;
pp = ParametricPlot[{r, k}, {x, 1.001, 20}, AspectRatio -> 1,
   PlotRange -> {{0, 40}, {0, 0.8}}, AxesLabel -> {"r", "k"}];
te = Graphics[{Text[Style["refuge", 12], {10, .2}],
    Text[Style["bistable", 12], {20, .4}],
    Text[Style["outbreak", 12], {20, .6}]}];
Show[pp, te, ImageSize -> Small]
```

Chapter 3 Homework

- 3.1: 1, 3
- 3.2: 1, 4
- 3.4: 1, 3, 7, 8
- 3.7: 1, 2

Chapter 5. Two dimensional linear systems

5.1. Definitions and Examples

A **two dimensional (autonomous) linear system** is of the form

$$\begin{aligned}\dot{x}(t) &= ax(t) + by(t) \\ \dot{y}(t) &= cx(t) + dy(t)\end{aligned} \tag{11}$$

where a, b, c, d are parameters. Letting

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

this system can be written in the compact form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

This system has the special property of **superposition principle**. Namely, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two solutions then so is $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ for any choice of constants c_1, c_2.

Equilibrium solutions of this system is given by the set of solutions of

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

For linear systems (11), the origin

$$\mathbf{x}^* = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is always an equilibrium solution which is sometimes called the **trivial equilibrium**. Basic linear algebra says that no other equilibria exists if $\det A \neq 0$. On the other hand, if $\det(A) = 0$ then there are infinitely many equilibria.

A solution $(x(t), y(t))$ can be visualized as a **trajectory** in the xy-plane, called the **phase plane**, which is always tangent to the vector field $(ax + by, cx + dy)$. The lines $ax + by = 0$ and $cx + dy = 0$ are called **nullclines** of this vector field. The intersection of the nullclines yield the equilibria, one of which is always located at the origin as discussed.

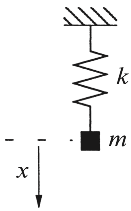


Figure 22: Mass hanging from a linear spring.

Example 18. *According to Hooke's Law*[1]*, the mass hanging from a linear spring is modeled by*

$$m\ddot{x} + kx = 0$$

where $k > 0$ is a characteristic of the spring, and $m > 0$ is the mass. See Figure 22.

We convert this equation to a 2d system by writing

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x. \end{aligned}$$

[1]Hooke's Law is an accurate approximation as long as the forces and deformations are small enough.

where $\omega^2 = k/m$. The vector plot Figure 23 and the stream plot Figure 24 for $\omega = 1.5$ are produced by Code 2.

*The only fixed point is the origin which corresponds to mass sitting at the equilibrium forever. Besides there are infinitely many **closed trajectories** around the origin. These correspond to time-periodic motions of the spring. See Figure 25.*

Code 2.

```
w = 1.5;
VectorPlot[{v, -w^2*x}, {x, -1, 1}, {v, -1, 1}, Axes -> True,
 AxesLabel -> {x, v}]
StreamPlot[{v, -w^2*x}, {x, -1, 1}, {v, -1, 1}, Axes -> True,
 AxesLabel -> {x, v}]
```

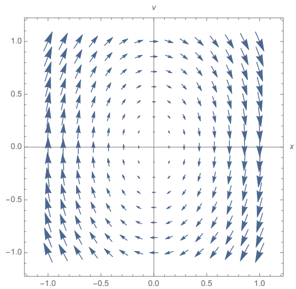


Figure 23: The vector plot for $(v, -\omega^2 x)$.

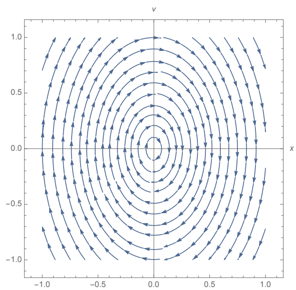


Figure 24: The stream plot for $(v, -\omega^2 x)$.

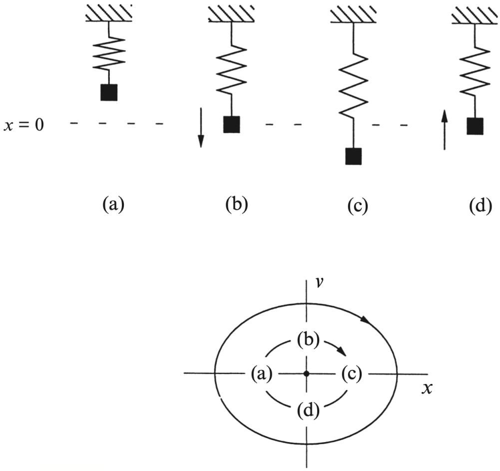


Figure 25: Closed trajectories correspond to time periodic motions of the spring.

Example 19. *Sketch the phase portrait of $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$ for different a showing qualitatively different cases.*

Since the equations are uncoupled, the solution is easy to obtain

$$x(t) = x_0 e^{at}, \qquad y(t) = y_0 e^{-t}$$

From this, we can obtain

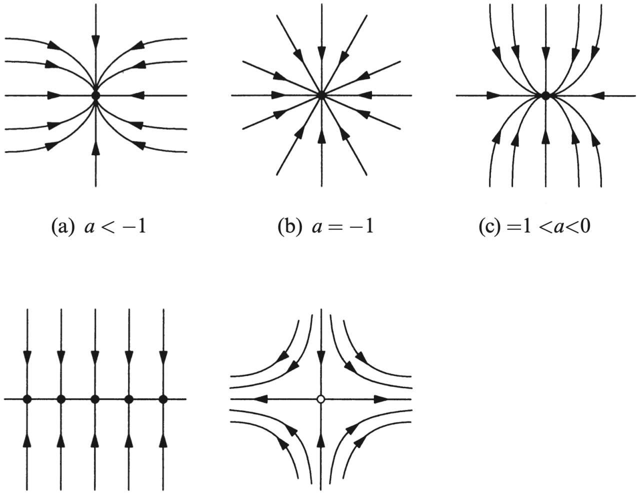
$$y^{-a} = y_0^{-a} e^{at} = \frac{y_0^{-a}}{x_0} x$$

or

$$x = \frac{c}{y^a}$$

*From this we can obtain Figure 26. When $a < 0$, we will call the origin a **stable node**, when $a > 0$ we will call it a **saddle**. The case $a = 0$ is a degenerate case, where each point on the x-axis is an equilibrium.*

*When the origin is a saddle, there are two important manifolds. The **stable manifold** is the set of initial conditions $\mathbf{x}_0$ such that $x(t) \to \mathbf{x}^* = \mathbf{0}$ as $t \to \infty$. For this example the stable manifold of the origin is the y-axis. Similarly one defines **unstable manifold** as the set of initial conditions $\mathbf{x}_0$ for which $x(t) \to \mathbf{0}$ as $t \to -\infty$. The stable manifold of the origin is the x-axis.*



(a) $a < -1$ (b) $a = -1$ (c) $=1 < a < 0$

(d) $a = 0$ (e) $a > 0$

Figure 26:

Stability Language

Let $\mathbf{x}^*$ be an equilibrium of an autonomous dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

1. $\mathbf{x}^*$ is said to be **Lyapunov stable** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ then $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$, for every $t \geq 0$.

2. $\mathbf{x}^*$ is said to be **attracting** if there exists $\delta > 0$ such that if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ then $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}^*\| = 0$.

3. $\mathbf{x}^*$ is said to be **asymptotically stable** if it is Lyapunov stable and attracting.

4. $\mathbf{x}^*$ is said to be **neutrally stable** if it is Lyapunov stable but not attracting.

The origin is Lyapunov stable Figure 26a-d, asymptotically stable in Figure 26a-c, neutrally stable in Figure 26d.

It is possible for an equilibrium to be attracting while not being Lyapunov stable, see Figure 27.

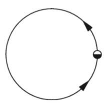


Figure 27: For $\dot{\theta} = 1 - \cos\theta$, $\theta^* = 0$ is an attracting equilibrium which is not Lyapunov stable.

5.2. Classification of Linear Systems

To find the general solution of (11), we assume a solution of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

Plugging this into the equation, we get

$$A\mathbf{x} = \lambda \mathbf{x}$$

This means that λ is an eigenvalue and $\mathbf{v}$ is the corresponding eigenvector.

Recall that the eigenvalues can be found by the roots of the characteristic equation

$$0 = \det(A - \lambda I) = \lambda^2 - \tau\lambda + \Delta$$

where $\tau = \mathrm{tr}(A) = a + d$ and $\Delta = \det A = ad - bc$. The eigenvalues are thus given by

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

The case $\tau^2 - 4\Delta > 0$

In this case the eigenvalues λ_1, λ_2 are distinct and real. From linear algebra we know that the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to distinct eigenvalues are linearly independent and thus span $\mathbb{R}^2$. Hence the initial condition can be written as a linear combination

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some c_1, c_2. Since the equation is linear, the superposition of two solutions is also a solution. Hence

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

is a solution. It also satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. By uniqueness theorem, it is the only solution.

If the initial condition $\mathbf{x}_0$ lies on the direction of $\mathbf{v}_1$ then $c_2 = 0$. In this case $x(t)$ lies in the direction of $\mathbf{v}_1$ for all t. If $\lambda_1 < 0$, this straight line orbit is towards the origin. If $\lambda_1 > 0$, this straight line orbit is away from the origin.

The case $\lambda_1 < 0 < \lambda_2$.

For any initial condition not on the eigen-directions, $\mathbf{v}_1$ component of the solution vanishes and the solution tends to infinity along the direction $\mathbf{v}_2$. The origin is called a **saddle point** and is unstable.

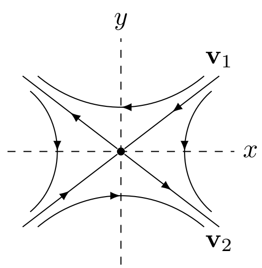


Figure 28: $\lambda_1 < 0 < \lambda_2$. The origin is called a saddle point.

The case $\lambda_1 < \lambda_2 < 0$.

In this case, since $e^{\lambda_1 t} \ll e^{\lambda_2 t}$ for large t, the $\mathbf{v}_1$ component of the solution decays faster than the $\mathbf{v}_2$ component of the solution. All the orbits tend to the origin tangent to the slow direction $\mathbf{v}_2$ as $t \to \infty$. $\mathbf{v}_1$ direction is called the **fast direction** and $\mathbf{v}_2$ direction is called the **slow direction**. The origin is called a **stable node** and is asymptotically stable.

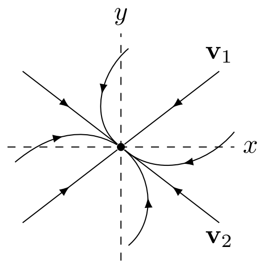


Figure 29: $\lambda_1 < \lambda_2 < 0$. The origin is called a stable node.

The case $0 < \lambda_1 < \lambda_2$.

In this case the orbits tend to infinity parallel to the fast direction $\mathbf{v}_2$ as $t \to \infty$. Similarly they tend to the origin tangent to the slow direction $\mathbf{v}_1$ as $t \to -\infty$. The origin is called an **unstable node**.

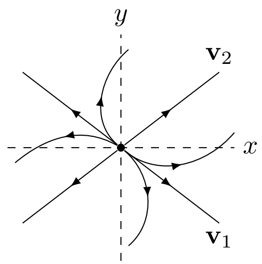


Figure 30: $0 < \lambda_1 < \lambda_2$. The origin is called a stable node.

The case $\lambda_1 < \lambda_2 = 0$.

In this case, the solution becomes

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2\mathbf{v}_2$$

If the $\mathbf{v}_1$ part of the initial condition is zero then $c_1 = 0$ and the solution is an equilibrium solution $x(t) = c_2\mathbf{v}_2$. Thus any point in the direction of $\mathbf{v}_2$ is an equilibrium. In other words, there are infinitely equilibria. The origin (and every other equilibria) are neutrally stable.

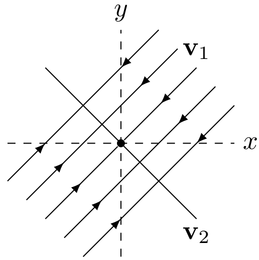


Figure 31: $\lambda_1 < \lambda_2 = 0$. There are infinitely many equilibria along the direction of $\mathbf{v}_2$. The origin (and every other equilibria) are neutrally stable.

The case $0 = \lambda_1 < \lambda_2$. Similar to the above case but reverse in direction.

The case $\tau^2 - 4\Delta < 0$

In this case, the eigenvalues are $\lambda_1 = \overline{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_1 = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Also the eigenvectors are $\mathbf{v}_1 = \overline{\mathbf{v}_2} = \mathbf{a} + i\mathbf{b}$. Then by Euler's formula

$$e^{\lambda_1 t}\mathbf{v}_1 = e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)(\mathbf{a} + i\mathbf{b}) = \mathbf{u}(t) + i\mathbf{v}(t)$$

where

$$\begin{aligned}\mathbf{u}(t) &= e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)\\ \mathbf{v}(t) &= e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)\end{aligned}$$

It can be shown that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent solutions. The general solution can be written as

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) = e^{\alpha t}\left(\mathbf{A}\cos\beta t + \mathbf{B}\sin\beta t\right)$$

where

$$\begin{aligned}\mathbf{A} =& (c_1\mathbf{a} + c_2\mathbf{b})\\ \mathbf{B} =& (c_1\mathbf{a} - c_2\mathbf{b})\end{aligned}$$

If $\alpha < 0$ then the solutions spiral towards the origin as $t \to \infty$. The origin is called a **stable spiral** and is asymptotically stable.

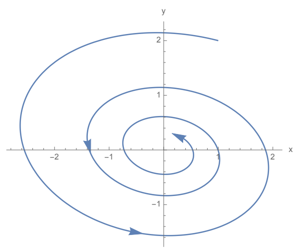


Figure 32: The origin is a stable focus if the real part of $\lambda_1 < 0$. See Listing 4 for the code.

Listing 4: Code for Figure 32

```
a = {1, 2};
b = {-3, 1};
c = -0.1;
d = 1;
ParametricPlot[{Exp[c*t] (a*Cos[d*t] + b*Sin[d*t])}, {t, 0, 6 \[Pi]},
  PlotRange -> Full, TicksStyle -> Directive[FontSize -> 16],
  AxesLabel -> {Style["x", Bold, 16], Style["y", Bold, 16]}] /.
 Line[x_] :> {Arrowheads[{0., 0.05, 0.05, 0.05}], Arrow[x]}
```

If $\alpha > 0$ then the solutions spiral away from the origin to infinity. The origin is called an **unstable spiral**.

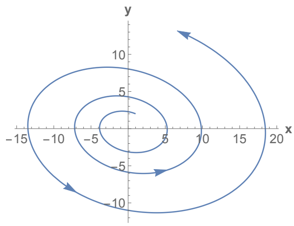


Figure 33: The origin is an unstable focus if the real part of $\lambda_1 > 0$.

If $\alpha = 0$, then the solutions are given by

$$\mathbf{x}(t) = \mathbf{A}\cos\beta t + \mathbf{B}\sin\beta t$$

Note that if $\mathbf{x}(0) = \mathbf{A}$. Moreover, if $\mathbf{A} \neq \mathbf{0}$ then The solutions with initial condition $\mathbf{x}_0 \neq \mathbf{0}$ are **periodic** with period $2\pi/\beta$. The closed orbits are ellipses. The origin is called a **center**.

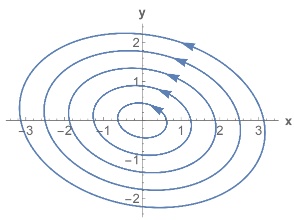


Figure 34: The origin is a center if the real part of $\lambda_1 = 0$. Here the orbits corresponding to five different initial conditions are shown.

5.3. Love Affairs: To do

Chapter 5 Homework

1. 5.1: 1, 2, 10ad.
2. 5.2: 1, 2, 3, 6, 9, 11.

Chapter 6. Two dimensional nonlinear systems

6.1 Phase Portraits

Watch: https://www.youtube.com/watch?v=9yh9DmNqdk4

We consider systems of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{12}$$

or by defining $\mathbf{x} = (x_1, x_2)$, $\mathbf{f} = (f_1, f_2)$ we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

We can also consider an initial point

$$\mathbf{x}(0) = \mathbf{x}_0$$

For each initial point $\mathbf{x}_0$, suppose there is a solution (existence and uniqueness theorem) which we denote by

$$\mathbf{x}(t; \mathbf{x}_0)$$

For each $\mathbf{x}_0$, the curves

$$t \to \mathbf{x}(t; \mathbf{x}_0)$$

is called an **integral curve** of (12). The parametric plot of these curves in the x-y plane is called the **phase portrait** of (12).

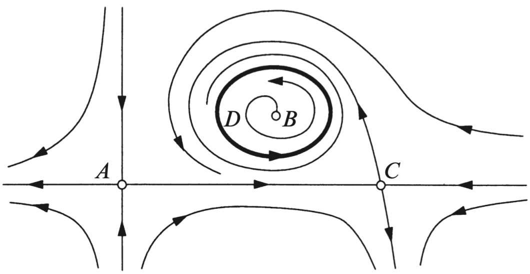


Figure 35: A typical phase portrait.

Some interesting features of a typical phase portrait such as Figure 35

1. If $f(\mathbf{x}^*) = \mathbf{0}$ then $\mathbf{x}(t, \mathbf{x}^*) = \mathbf{x}^*$ for all $t \in \mathbb{R}$. The **fixed points** like A, B, C where $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$ which correspond to **steady state solutions** also known as **equilibrium solutions**. These solutions are independent of time, that is they are constant.

2. The **closed orbits** like D which correspond to periodic solutions, i.e. solutions with $\mathbf{x}(t$

Listing 5: Code for Figure 37

```
plot1 = StreamPlot[{x + Exp[-y], -y}, {x, -3, 2}, {y, -2, 3}];
plot2 = ParametricPlot[{-Exp[-y], y}, {y, -2, 3}, PlotStyle -> {Red, Dashed}];
Show[plot1, plot2, Axes -> True, AxesStyle -> Directive[Thick, Dashed, Gray]]
```

Example 20 (Example 6.1.1)**.**

$$\begin{aligned} \dot{x} &= x + e^{-y} \\ \dot{y} &= -y \end{aligned} \tag{13}$$

Sketch the phase portrait.

Solution. The only fixed point is $(-1, 0)$. To determine its stability, note that $y(t) \to 0$ as $t \to \infty$ and for large t, $\dot{x} \approx x + 1$ which has an exponentially growing solution and the fixed point is unstable.

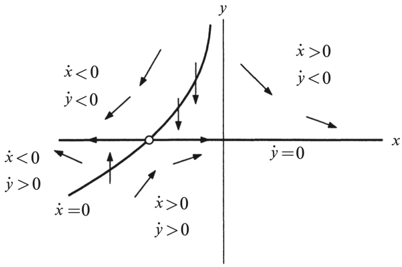


Figure 36: Nullclines of (13).

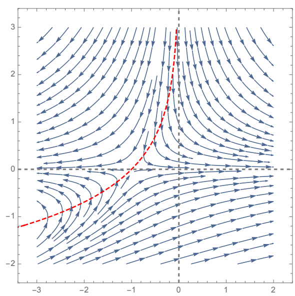


Figure 37: Phase portrait of (13). Red dashed contour is $x + e^{-y} = 0$ which is not an integral curve

6.3. Fixed Points and Linearization

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

Suppose (x^*, y^*) is a fixed point, that is

$$f(x^*, y^*) = g(x^*, y^*) = 0$$

Consider small deviations,

$$u(t) = x(t) - x^*, \qquad v(t) = y(t) - y^*$$

$$\dot{u} = \dot{x} = f(x, y) = f(x^* + u, y^* + v) = f(x^*, y^*) + u\frac{\partial f}{\partial x}(x^*, y^*) + v\frac{\partial f}{\partial y}(x^*, y^*) + \text{h.o.t.}$$

$$\dot{v} = \dot{x} = g(x, y) = g(x^* + u, y^* + v) = g(x^*, y^*) + u\frac{\partial g}{\partial x}(x^*, y^*) + v\frac{\partial g}{\partial y}(x^*, y^*) + \text{h.o.t.}$$

Let the Jacobian matrix be

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Since $f(x^*, y^*) = g(x^*, y^*) = 0$, if we neglect h.o.t, we have linearization about x^*.

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Question. Does linearization give qualitatively correct dynamics near a fixed point x^*?

Answer. Yes if the real parts of the eigenvalues of the Jacobian matrix are non-zero. That is when $\mathbf{x}^*$ is a saddle, a node or a spiral. But borderline cases (degenerate node, star, center, non-isolated fixed point) can be altered by the nonlinear terms.

Here is an example where the center of linear system is altered by nonlinear terms.

Example 21.

$$\begin{aligned} \dot{x} &= -y + ax\left(x^2 + y^2\right) \\ \dot{y} &= x + ay\left(x^2 + y^2\right) \end{aligned}$$

Show that the linearized system incorrectly predicts that the origin is a center for all values of a, whereas in fact the origin is a stable spiral if $a < 0$ *and an unstable spiral if* $a > 0$.

Solution.

The Jacobian is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which has trace $\tau = 0$ *and determinant* $\Delta = 1 > 0$. *Thus the origin is a center.*

To analyze the nonlinear system, we change variables to polar coordinates. Let $x = r\cos\theta$ *and* $y = r\sin\theta$.

$$x^2 + y^2 = r^2 \implies x\dot{x} + y\dot{y} = r\dot{r}$$

$$r\dot{r} = ar^4 \implies \dot{r} = ar^3$$

$$\theta = \arctan\left(\frac{y}{x}\right) \implies \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

$$\begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= 1 \end{aligned} \tag{14}$$

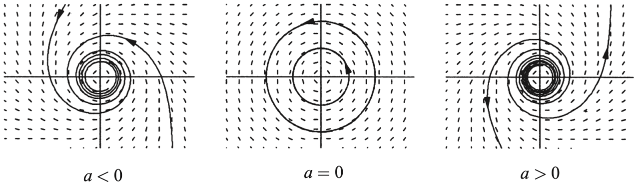


Figure 38: Behavior of (14)

6.4. Rabbits vs Sheep

Lotka-Volterra model of competition: $x(t)$ is the population of rabbits, $y(t)$ is the population of sheep. $x, y \geq 0$.

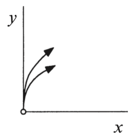
$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

If $y = 0$, rabbits grow as $\dot{x} = x(3 - x)$, this is logistic growth. The effect of sheep on rabbits is $-2xy$.

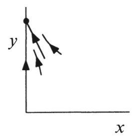
Fixed points: $(0,0)$, $(3,0)$, $(0,2)$, $(1,1)$. Jacobian matrix is

$$A = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}$$

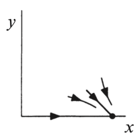
For $(0,0)$: $\lambda_1 = 3$, $\lambda_2 = 2$ unstable node. Trajectories leave along $\lambda_2 = 2$ direction $\mathbf{e}_2 = (0,1)^T$.



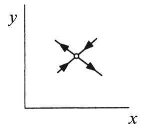
For $(0,2)$: $\lambda_1 = -1$, $\lambda_2 = -2$ stable node. Trajectories approach along $\lambda_1 = -1$ eigendirection which is $\begin{matrix}1\\-2\end{matrix}$



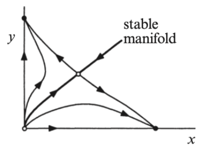
For $(3,0)$: $\lambda_1 = -3$, $\lambda_2 = -1$ stable node.



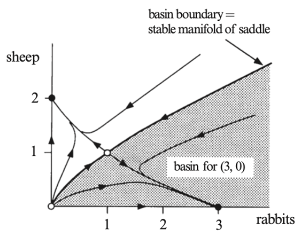
For $(1,1)$: $\tau = -2$, $\Delta = -1$, a saddle



Combining all together



The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other mod- els of competition and has led biologists to formulate the principle of competitive exclusion, which states that two species competing for the same limited resource typically cannot coexist.



6.5. Conservative Systems

Watch: https://www.youtube.com/watch?v=3s2lmZspEU8

Consider mechanical system with 1 degree of freedom

$$m\ddot{x} = F(x) = -\frac{dV}{dx}$$

$$m\ddot{x}\dot{x} + \frac{dV}{dx}\dot{x} = 0 \implies \frac{d}{dt}\left(\frac{1}{2}m\dot{x} + V(x)\right) = 0$$

Hence the total energy

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \equiv C$$

For $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $E(\mathbf{x})$ is a **conserved quantity** if $E(\mathbf{x})$ is a continuous, real valued function that is constant on trajectories, that is $\frac{dE}{dt} = 0$, and not identically constant on any open set (otherwise $E(\mathbf{x}) = 42$ is a conserved quantity). If a system has a conserved quantity, then it is called a **conservative system**.

Note that E is constant on trajectories mean that trajectories must lie on the level sets of E (sets on which E is constant).

Example 22. *A conservative system cannot have any attracting/repelling fixed points.*

Solution. *Suppose $\mathbf{x}^*$ were an attracting fixed point. Then $E(\mathbf{x}^*)$ would be constant on the basin of attraction of $\mathbf{x}^*$.*

Example 23. *Consider the double well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ with mass $m = 1$.*

$$\ddot{x} = x - x^3.$$

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

The equilibria are $(0,0)$, $(\pm 1, 0)$. The Jacobian is

$$A = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{bmatrix}$$

At $(0,0)$, $\Delta = -1$ which is a saddle. At $(\pm 1, 0)$, $\tau = 0$, $\Delta = 2$ which are centers. We know that small nonlinear terms can destroy centers predicted by the linear approximation. But here they are actually centers.

Recall $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$. Trajectories lie on the sets $E = c$, $c \in \mathbb{R}$. So we need plot the contours of $E = c$.

To sketch this graph one can use Mathematica:

```
ContourPlot[y^2/2 - x^2/2 + x^4/4,{x,-2, 2}, {y -2, 2}]
```

To manually sketch:

1. *Near $(0,0)$, ignore x^4 term,*

$$E \approx y^2/2 - x^2/2 = C$$

which are hyperbolas.

2. *Near $(1,0)$, $(x+1)^2 \approx 4$ and*

$$E = y^2/2 - 1/4(x-1)^2(x+1)^2 \approx y^2/2 - (x-1)^2 = C'$$

which are ellipses.

3. *The vector field is vertical, $\dot{x} = 0$ on the line $y = 0$. The vector field is horizontal, $\dot{y} = 0$ on the lines $x = -1$, $x = 0$, $x = 1$.*

4. *The contours have symmetries $x \to -x$ and $y \to -y$. So the picture will be symmetric with respect to both x and y axes.*

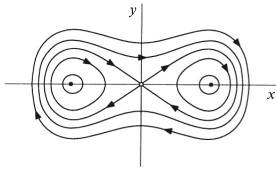


Figure 39: Contours of $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$

Thus solutions of the system are typically periodic, except for the equilibrium solutions and two very special trajectories: which start and end at $(0, 0)$ *called* ***homoclinic orbit****. They are common in conservative systems but are rare otherwise.*

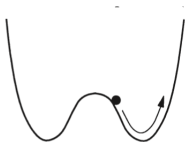


Figure 40: The graph of $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. Periodic orbits can be understood as the oscillations of the particle in the double well. The homoclinic orbit corresponds to the trajectory of a particle with just enough energy to end up (in infinite time) at the top.

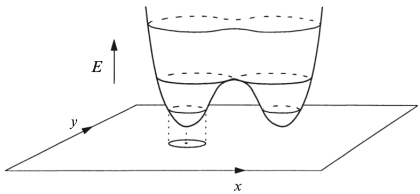


Figure 41: Graph of $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$. The local minima of E correspond to centers at $(\pm 1, 0)$ while the saddle of E corresponds to the saddle $(0, 0)$.

Theorem 4. *Let* $\dot{x} = f(x)$ *be a system with a conserved quantity* E*. At the local isolated minima/maxima of* E *(since the contours of* E *are closed by the Morse Lemma) the system has a fixed point which is center. At the isolated saddles of* E *the system has a saddle.*

Recall that the critical point of $E(x, y)$ are given by $\frac{\partial E}{\partial x} = \frac{\partial E}{\partial y} = 0$. Moreover let

$$\Delta E = \begin{vmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{vmatrix}$$

1. If $\Delta E > 0$ and $E_{xx} > 0$ at (x_0, y_0) then E has an isolated local minimum at (x_0, y_0).
2. If $\Delta E > 0$ and $E_{xx} < 0$ at (x_0, y_0) then E has an isolated local maximum at (x_0, y_0).
3. If $\Delta E < 0$ then E has an isolated saddle at (x_0, y_0).

6.7. Pendulum

Example 24. *Pendulum:* $\ddot{\theta}+\sin\theta=0$. *Let* $v=\dot{\theta}$. *This is a conservative system with potential* $-\sin\theta=-V'(\theta)$ *and* $V(\theta)=-\cos\theta$ *and energy* $E=v^2/2-\cos\theta$. *Critical points of* E *occur at* $\frac{\partial E}{\partial\theta}=0$, $\frac{\partial E}{\partial v}=0$ *which are* $\theta=0$, $\sin\theta=0$. *The critical points are* $(0,n\pi)$, $n\in\mathbb{Z}$.

$$\Delta E=\begin{vmatrix}\cos\theta & 0\\ 0 & 1\end{vmatrix}=\cos\theta$$

E *has a local minimum at* $(\theta,v)=(0,2n\pi)$, $n\in\mathbb{Z}$. E *has saddles at* $(\theta,\dot{\theta})=(0,\pi+2n\pi)$, $n\in\mathbb{Z}$.

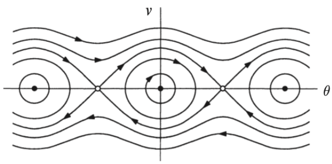
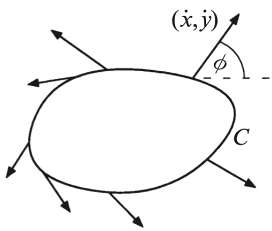


Figure 42: Graph of $E=\dot{\theta}^2/2-\cos\theta$.

6.8. Index Theory

https://www.youtube.com/watch?v=02fcpxLT5wk Index theory is a method that provides global information about the phase portrait.

Let $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^2$. Consider a simple (not intersecting itself) closed curve C which does not pass through any fixed points of $\mathbf{f}$. Note that C is not necessarily a trajectory. At each point on C, we can define the angle $\phi=\arctan(\dot{y}/\dot{x})$.



The **index of the closed curve** with respect to the vector field $\mathbf{f}$ is

$$I_C=\frac{1}{2\pi}[\phi]_C$$

where $[\phi]_C$ is the net change in angle as the curve is transversed counterclockwise. Note that I_C must be an integer which is the number of net revolutions.

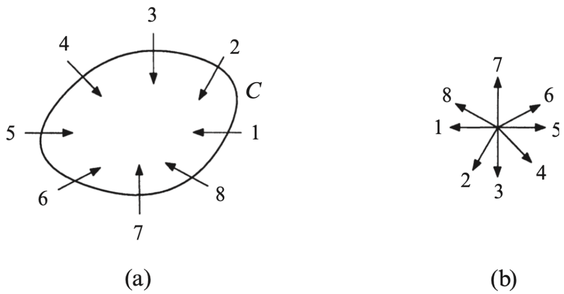


Figure 43: $I_C = 1$.

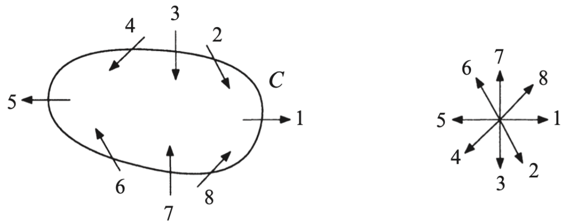


Figure 44: $I_C = -1$.

Example 25. *Given $\dot{x} = x^2 y$, $\dot{y} = x^2 - y^2$ find I_C where C is the unit circle.*

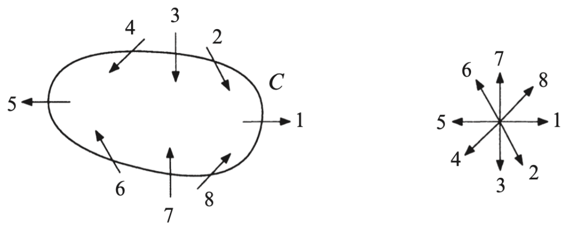


Figure 45: $I_C = 0$.

Properties of the Index

1. If C can be continuously deformed into C' without passing through a fixed point then $I_C = I_{C'}$. proof. I_C changes continuously and is an integer. It must be constant.

2. If C does not enclose any fixed points then $I_C = 0$. proof. By the previous property, shrink C to a tiny circle C' without changing the index. But $I_{C'} = 0$ since the vector field is almost constant on C'.

3. If we reverse the arrows by $t \to -t$ the index is unchanged. proof. All angles change from ϕ to $\phi + \pi$ and the net change in ϕ stay the same.

4. Index of a closed trajectory = +1.

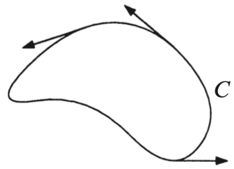


Figure 46: Index of a closed trajectory = +1.

Index of a Point Suppose $\mathbf{x}^*$ is an isolated fixed point of $\mathbf{f}$. Then the index of I of $\mathbf{x}^*$ is defined as I_C where C is any closed curve that encloses $\mathbf{x}^*$ and no other fixed points. By (1) above, index of a point is well-defined.

Index of a stable node is 1. By property (3) index of an unstable node is also 1. Index of a saddle is -1. Index of a non fixed point is 0.

Spirals, centers, degenerate nodes and stars all have index 1. Thus, a saddle point is truly a different animal from all the other familiar types of isolated fixed points.

Theorem 5. *Any closed trajectory on $\mathbb{R}^2$ must enclose at least one fixed point. Moreover if it encloses n fixed points with index I_i, then*

$$I_C = I_1 + I_2 + \cdots I_n$$

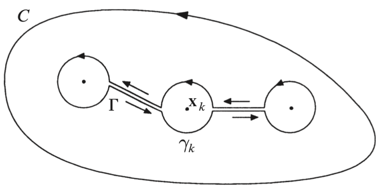


Figure 47: Idea of the proof.

Proof. □

We can use index theory to rule out closed trajectories.

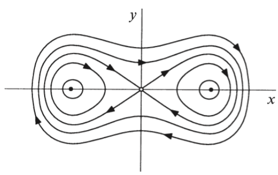


Figure 48: A closed trajectory in the plane can enclose two centers and a saddle but not two saddles and a center.

Example 26. *Rabbit vs sheep system.*

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

with $x, y \geq 0$. *As shown before there are 4 fixed points* $(0,0)$ *unstable node;* $(0,2)$ *and* $(3,0)$ *stable nodes;* $(1,1)$ *saddle point. There can be no closed trajectory for this system.*

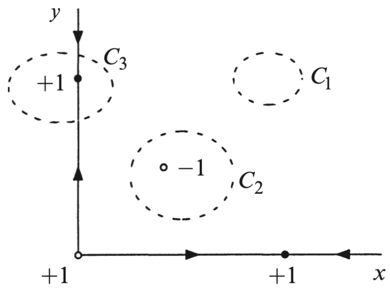


Figure 49: Rabbit sheep system can not have any closed trajectory.

Example 27. *Show that the system* $\dot{x} = xe^{-x}$, $\dot{y} = 1 + x + y^2$ *has no closed orbits.* Solution. *This system has no fixed points so it can not have any closed orbits.*

Index theory can be generalized to 2-manifolds.

Remark 1. *A neighborhood of an equilibrium point consists of the following sectors:*

1. *Elliptic sectors filled with orbits starting and ending at the equilibrium.*
2. *Hyperbolic sectors filled with orbits roughly resembling hyperbolas.*
3. *Parabolic sectors filled with orbits having only one end at the equilibrium.*

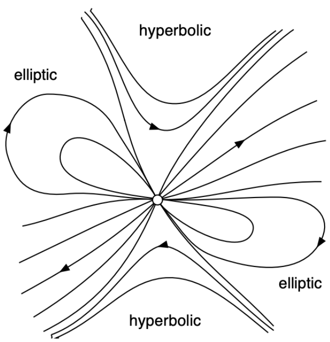


Figure 50: Sectors of a neighborhood of an equilibrium.

Bendixon's formula states that the index of an equilibrium is

$$1 + \frac{e-h}{2}$$

where e is the number of elliptic sectors and h is the number of hyperbolic sectors. For proof see [2].

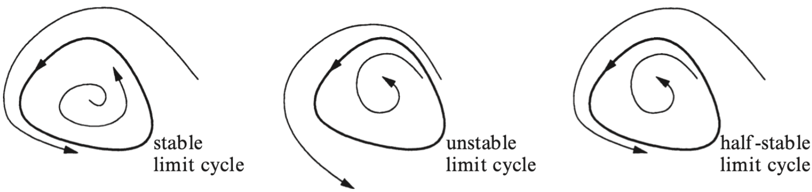
Chapter 6 Homework

1. 6.1: 1, 3, 14
2. 6.3: 1, 3, 12, 13, 15
3. 6.4: 1, 2, 3
4. 6.8: 1, 2, 3, 4, 5, 7, 11, 13

Chapter 7. Limit Cycles.

7.0. Introduction

A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.



Theorem. Linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ can not have limit cycles. **Proof**. They can have non-isolated periodic orbits. Because if $\mathbf{x}(t)$ is a closed trajectory then so is $c\mathbf{x}(t)$ for every c.

Example 28. *Consider $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ where $r \geq 0$. The system has a limit cycle $x(t) = \cos(t+\theta_0)$ and $y(t) = \sin(t+\theta_0)$*

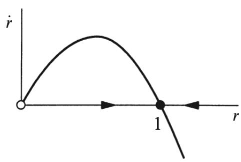


Figure 51: Stability of $\dot{r} = r(1-r^2)$.

[2] https://docs.univr.it/documenti/OccorrenzaIns/matdid/matdid940050.pdf

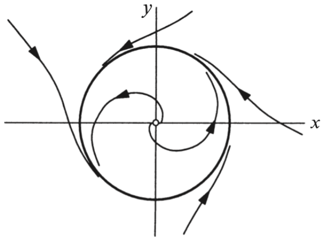


Figure 52: Stable limit cycle of $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$.

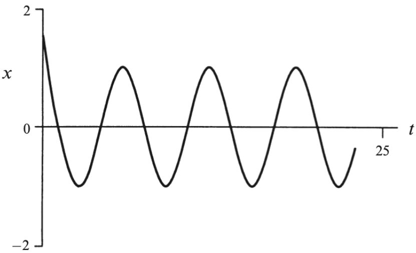


Figure 53: x coordinate of the solution of $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ with some initial condition.

7.2 Ruling Out Closed Orbits

Ways to rule out closed orbit (closed trajectory):

Index theory: if there is a closed orbit then the sum of indices of fixed points inside must be +1. See the example: rabbit vs sheep system.

Gradient systems: Closed orbits are impossible in gradient systems $\dot{\mathbf{x}} = -\nabla V$. Because on a trajectory V is constant. So if C is closed trajectory corresponding to the periodic solution $x(t + T) = x(t)$ with $T > 0$ then

$$0 = V(x(T)) - V(x(0)) = \int_0^T \frac{dV(\mathbf{x})}{dt} dt = \int_0^T \nabla V \cdot \dot{\mathbf{x}} dt = \int_0^T - \|\dot{\mathbf{x}}\|^2 \, dt < 0$$

unless $\dot{\mathbf{x}}(t) = 0$ for all $0 < t < T$ which implies the orbit is a fixed point and not a closed trajectory.

Example 29. *There are no closed orbits for $\dot{x} = \sin y$, $\dot{y} = x \cos y$. Solution. The system is a gradient system with potential $V = -x \sin y$.*

Liapunov Functions Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose

1. L is positive definite. $L(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ and $L(\mathbf{x}^*) = 0$.
2. $\dot{L}$ is negative definite. $\dot{L} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$.

Then $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditions $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In particular the system has no closed orbits.

For the proof:

$$L(\mathbf{x}(t)) = L(\mathbf{x}(0)) + \int_0^t \dot{L}(\mathbf{x}(s))ds < L(\mathbf{x}(0)).$$

So $L(\mathbf{x}(t))$ is decreasing and minimum of $L(\mathbf{x})$ is at $\mathbf{x} = 0$. The actual proof is a bit more involved.

Example 30. *Show that the system $\dot{x} = -x + 4y$ and $\dot{y} = -x - y^3$ has no closed orbits by using a Liapunov function of the form $L(x, y) = x^2 + ay^2$, choosing a carefully.*

Solution. Note that $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. $\dot{V} = -2x^2 + (8 - 2a)xy - 2ay^4$ which is negative-definite if $a = 4$.

Disadvantage of the Liapunov method. No general way of finding a Liapunov function.

Dulac's criterion. Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a smooth vector field on a **simply connected** (no holes inside) subset R of the plane. If there exists a smooth real valued function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign on R then there are no closed orbits lying in R.

Proof. If C is a closed trajectory and $\mathbf{n}$ be its normal vector. Then $\oint_C g\dot{\mathbf{x}} \cdot \mathbf{n} = 0$ since $\dot{\mathbf{x}} \cdot \mathbf{n} = 0$ on C. Let A be the region enclosed by C. By Green's Theorem

$$\iint_A \nabla \cdot (g\dot{\mathbf{x}})dA = \oint_C g\dot{\mathbf{x}} \cdot \mathbf{n} = 0$$

which is a contradiction since $\nabla \cdot (g\dot{\mathbf{x}})$ has a single sign.

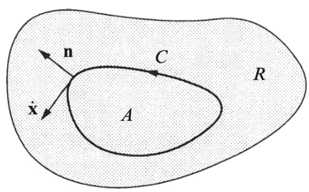


Figure 54: Proof of Dulac's criterion.

□

Example 31. *Show $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ has no closed orbits on $x, y > 0$.*

Solution. Pick $g = 1/xy$. See $\nabla \cdot (g\dot{\mathbf{x}}) < 0$ and the domain is simply-connected.

Disadvantage of the Dulac's method.

No general method for finding g. It is hard to guess a g.

Try $g = 1, 1/(xy), 1/(x^a y^b)$, ...

Example 32. *Study example 7.2.5.*

7.3 Poincaré-Bendixson Theorem

In this section we show methods to show the existence of closed orbits.

Poincaré-Bendixson Theorem. Suppose

1. R is closed, bounded region in $\mathbb{R}^2$.
2. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is smooth.
3. no fixed points in R.
4. there exists a trapped trajectory C: it starts in R and stays in R for all t.

Then either C is a closed trajectory or it spirals to a closed trajectory as $t \to \infty$.

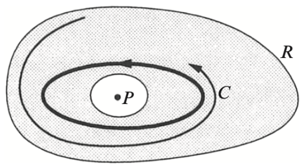


Figure 55: Region R in Poincaré-Bendixson Theorem will generally look like this because a closed trajectoy always encloseds a fixed point in the plane by the index theory.

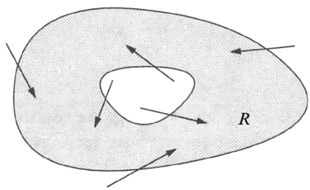


Figure 56: The key to finding a trapped trajectory is to construct a **trapping region** R where no trajectory can get out and all trajectories in R are trapped.

Example 33. *Consider*

$$\dot{r} = r(1 - r^2) + \mu r \cos\theta$$
$$\dot{\theta} = 1$$

Show that a closed orbit exists for all $0 < \mu < 1$.

Solution. Fix μ. *Construct a trapping region by seeking two concentric circles with radii* $r_{min} > 0$ *and* $r_{max} > 0$ *such* $\dot{r} < 0$ *at* $r = r_{max}$ *and* $\dot{r} > 0$ *at* $r = r_{min}$. *This guarantees that the annulus* $r_{min} < r < r_{max}$ *is a trapping region.*

Notice that when $\mu = 0$, $\dot{r} < 0$ *if* $r > 1$ *and* $\dot{r} > 0$ *if* $r < 1$. *Since*

$$r(1 - r^2 - \mu) < \dot{r} < r(1 - r^2 + \mu)$$

choose

$$1 - r_{min}^2 - \mu > 0 \implies 0 < r_{min}^2 < 1 - \mu \implies 0 < r_{min} < \sqrt{1-\mu}$$

and choose

$$1 - r_{max}^2 + \mu < 0 \implies r_{max} > \sqrt{1+\mu}.$$

The result follows by Poincaré-Bendixson.

Example 34. *Glycolosis (breaking down of sugar to get energy).*

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

This is kinetic equation of glycolosis. x and y are concentrations of ADP and F6P, $a, b > 0$.

Solution. Construct a trapping region by using nullclines (curves where $\dot{x} = 0$, $\dot{y} = 0$.)

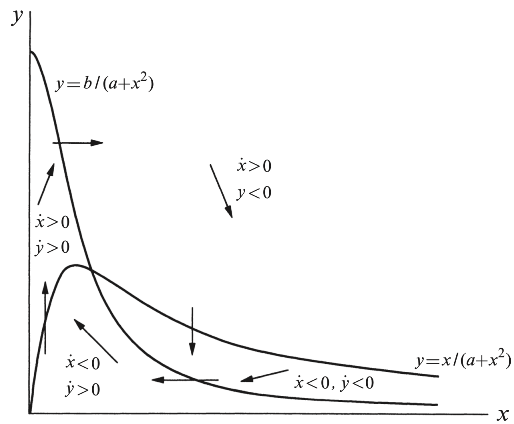


Figure 57: Nullclines of glycolosis system.

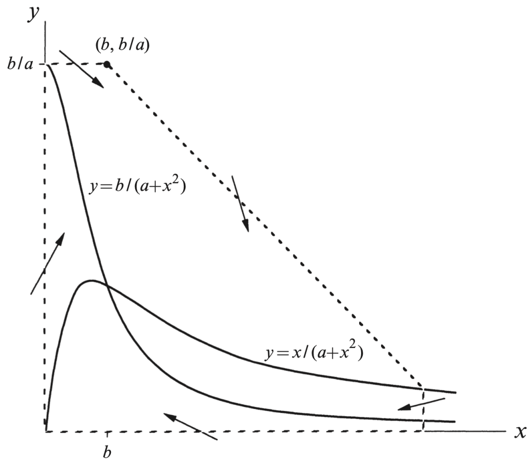


Figure 58: Trapping region of glycolosis system. The right boundary has slope -1.

How to come up with the right upper vertex $(b, b/a)$ *of the trappring region? First note that*

$$x, y \gg 1 \implies \dot{x} \approx x^2 y, \qquad \dot{y} \approx -x^2 y, \qquad \frac{\dot{y}}{\dot{x}} \approx -1$$

To better see

$$\dot{x} + \dot{y} = b - x \implies \frac{\dot{y}}{\dot{x}} < -1 \text{ if } x > b.$$

Can we conclude that there is a closed orbit inside the trapping region? No! There is a fixed point in the region (at the intersection of the nullclines), and so the conditions of the Poincaré-Bendixson theorem are not satisfied. But if this fixed point is a repeller, then we can prove the existence of a closed orbit by considering the modified "punctured" region. Do a linear stability analysis to find that at the fixed point, the determinant is $\Delta = a + b^2 > 0$ *and the trace is* $\tau > 0$.

No Chaos in the Phase Plane by Poincaré-Bendixson

Dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible.

In higher-dimensional systems, the Poincaré-Bendixson theorem no longer applies. It may happen that trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit.

7.5 Relaxation Oscillations

Mechanical and Electrical Vibrations

$$my'' = -\mu y' - ky$$

This equation models both mechanical vibrations such as motion of a spring and electrical vibrations in an electric circuit. In the case of mechanic spring, this follows from the Newton's Law. my'' is the acceleration term, $-ky$ is the restoring force due to the spring, $-\mu y'$ term is the **damping term**.

Show the following:

1. Without damping $\mu = 0$, this is just harmonic oscillator: all orbits are periodic except when $y(0) = y'(0) = 0$.
2. If $\mu > 0$, then all solutions decay to zero.
3. If $\mu < 0$ then all solutions tend to infinity as $t\ to\infty$.

The Van der Pol oscillator

The Van der Pol oscillator was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol while he was working at Philips. Van der Pol found stable oscillations,[2] which he subsequently called relaxation-oscillations and are now known as a type of limit cycle in electrical circuits employing vacuum tubes.

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$$

1. Nonlinear damping: $\mu\left(x^2 - 1\right)\dot{x}$. For $\mu > 0$, if $x^2 - 1 > 0$ then the damping is positive and the solutions decay, when $x^2 - 1 < 0$ then the damping is negative and the solutions are pumped. So it is plausible that the system will settle down to some oscillation.
2. Using Poincaré-Bendixson type argument, we can show that there exists a unique stable limit cycle for all $\mu > 0$. The proof is elobarate.

Aim is to investigate: $\mu \gg 1$ (in this section), $0 < \mu \ll 1$ (in the next section).
Trick: tricky change of variables.
Note:

$$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}\left(\dot{x} + \mu\left(\frac{1}{3}x^3 - x\right)\right)$$

Let

$$w = \dot{x} + \mu F(x), \qquad F(x) = \frac{1}{3}x^3 - x$$

From the equation

$$\begin{aligned}\dot{x} &= w - \mu F(x)\\ \dot{w} &= -x\end{aligned}$$

Let

$$y = \frac{w}{\mu}$$

Then

$$\begin{aligned}\dot{x} &= \mu(y - F(x))\\ \dot{y} &= -\frac{1}{\mu}x\end{aligned}$$

For $\mu \gg 1$,

$$y - F(x) \sim O(1) \implies |\dot{x}| \sim O(\mu) \gg 1, \qquad |\dot{y}| \sim O(\mu^{-1}) \ll 1.$$

Hence the velocity is enormous in the horizontal direction except on the cubic null-cline.

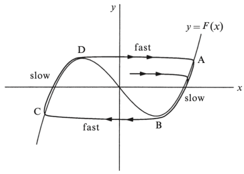
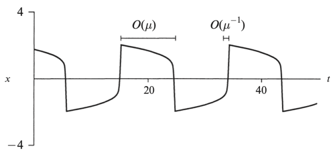
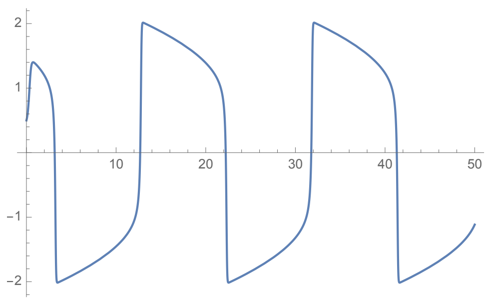


Figure 59: Van der Pol system. The graph has a local min at $B = (1, -\frac{2}{3})$ and local max at $D = (-1, \frac{2}{3})$. Compute A by $\frac{x^3}{3} - x = 2$ which gives $x = 2$. So $A = (2, \frac{2}{3})$. Similarly $C = (-2, -\frac{2}{3})$.

This analysis shows that the limit cycle has two widely separated time scales: the crawls require $\Delta t \sim O(\mu)$. Why? Intution. On the crawls, y-speed is $O(1/\mu)$ and the distance traveled is $O(1)$. So the time = distance/speed = $O(\mu) \gg 1$. The jumps require $\Delta t \sim O(\mu^{-1})$.






```
mu = 10;
sol = NDSolve[{x''[t] + mu*(x[t]^2 - 1) x'[t] + x[t] == 0, x[0] == .5,
     x'[0] == .5}, x, {t, 0, 5*mu}];
Plot[x[t] /. sol, {t, 0, 5*mu}]
```

Example 7.5.2 Estimate the period for $\mu \gg 1$.
The period $T \sim 2\times$(time from A to B).

Between A and B, use the fact that $w \approx \mu F(x)$.

$$T = 2\int_{t_A}^{t_B} dt \approx 2\int_2^1 \frac{dt}{dw}\frac{dw}{dx}dx = 2\int_2^1 \left(\frac{-1}{x}\right)\left(\mu(x^2-1)\right)dx = \mu\,(3 - 2\ln 2)$$

which is $O(\mu)$ as expected.

The formula can be refined. With much work, one can show that

$$T \approx \mu\,(3 - 2\ln 2) + 2\alpha\mu^{-1/3}$$

where $\alpha \approx 2.338\ldots$ is the smallest root of the Airy function.

Chapter 7 Homework

1. 7.1: 1, 3, 8.
2. 7.2: 1, 2, 6, 10, 12, 18
3. 7.3: 1, 3, 4, 5, 10.